Behaviour Properties of Uniformly Parameterised Cooperations

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Behaviour Properties of Uniformly Parameterised Cooperations

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In this paper we consider safety and liveness properties, where possibilistic aspects of especially liveness properties are captured by a modified satisfaction relation, called approximate satisfaction. The systems in focus of this paper are uniformly parameterised cooperations. Such systems are characterised by the composition of a set of identical components. These components interact in a uniform manner described by the schedules of the partners. Such kind of interaction is typical for scalable complex systems with cloud or grid structure. As a main result, a finite state verification framework for uniformly parameterised behaviour properties is given. The keys to this framework are structuring cooperations into phases and defining closed behaviours of systems. Finite state semi-algorithms that are independent of the concrete parameter setting are presented to verify behaviour properties of such uniformly parameterised cooperations.
Key Words: safety properties; possibilistic liveness properties; approximate satisfaction; uniformly parameterised cooperations; uniformly parameterised behaviour properties; finite state verification independent of the parameter settings

1. INTRODUCTION

The systems in focus of this paper are *uniformly parameterised cooperations*. Such systems are characterised by (i) the composition of a set of identical components (copies of a two-sided cooperation) and (ii) that these components "interact" in a uniform manner (described by the schedules of the partners). Such kind of interaction is typical for scalable complex systems. As an example for such uniformly parameterised systems of cooperations, e-commerce protocols can be considered. In these protocols the two cooperation partners have to perform a certain kind of financial transactions. As such a protocol should work for several partners in the same manner, and the mechanism (schedule) to determine how one partner may be involved in several cooperations is the same for each partner, the cooperation is parameterised by the partners and the parameterisation should be uniform w.r.t. the partners.

As a main result of the work presented, a finite state verification framework for *uniformly parameterised behaviour properties* of cooperations is given. To capture possibilistic aspects of especially liveness properties a modified satisfaction relation is used. For safety properties this relation, which is called approximate satisfaction, is equivalent to the usual one. The keys to this framework are structuring cooperations into phases and defining closed behaviours of systems. In that framework "completion of phases strategies" and corresponding "success conditions" are

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formalised which produce finite state semi-algorithms that are independent of the concrete parameter setting. These algorithms are used to verify behaviour properties of uniformly parameterised cooperations under certain regularity restrictions.

The subsequent paper is structured as follows. In Sect. 2 uniform parameterisations of two-sided cooperations in terms of formal language theory is formalised and a kind of self-similarity is considered. In this self-similarity concept, when only actions of some selected partners are considered, the complex system of all partners behaves like the smaller subsystem of the selected partners. Section 3 introduces the concept of uniformly parameterised behaviour properties of cooperations. The concept of structuring cooperations into phases given in Sect. 4 enables completion of phases strategies which are presented in Sect. 5. Consistent with this, corresponding success conditions are formalised which produce finite state semi-algorithms to verify behaviour properties of uniformly parameterised cooperations.

2. PARAMETERISED COOPERATIONS

The behaviour L of a discrete system can be formally described by the set of its possible sequences of actions. Therefore $L \subset \Sigma^*$ holds where Σ is the set of all actions of the system, and Σ^* (free monoid over Σ) is the set of all finite sequences of elements of Σ (words), including the empty sequence denoted by ε. $\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}$. Subsets of Σ^* are called formal languages [Sakarovitch 2009]. Words can be composed: if u and v are words, then uv is also a word. This operation is called the *concatenation*; especially $\varepsilon u = u\varepsilon = u$. Concatenation of formal languages $U, V \subset \Sigma^*$ are defined by $UV := \{uv \in \Sigma^* | u \in U \text{ and } v \in V\}$. A word u is called a *prefix* of a word v if there is a word x such that $v = ux$. The set of all prefixes of a word u is denoted by $\text{pre}(u)$; $\varepsilon \in \text{pre}(u)$ holds for every word u. The set of possible continuations of a word $u \in L$ is formalised by the *left quotient* $u^{-1}(L) := \{x \in \Sigma^* | ux \in L\}$.

Infinite words over Σ are called ω-words [Perrin and Pin 2004]. The set of all infinite words over Σ is denoted Σ^ω. An ω-language L over Σ is a subset of Σ^ω. For $u \in \Sigma^*$ and $v \in \Sigma^\omega$ the *left concatenation* $uv \in \Sigma^\omega$ is defined. It is also defined for $U \subset \Sigma^*$ and $V \subset \Sigma^\omega$ by $UV := \{uv \in \Sigma^\omega | u \in U \text{ and } v \in V\}$.

For an ω-word w the prefix set is given by the formal language $\text{pre}(w)$ which contains every finite prefix of w. The prefix set of an ω-language $L \subset \Sigma^\omega$ is accordingly given by $\text{pre}(L) = \{u \in \Sigma^* |$ it exist $v \in \Sigma^\omega$ with $uv \in L\}$. For $M \subset \Sigma^*$ the ω*-power* $M^\omega \subset \Sigma^\omega$ is the set of all "infinite concatenations" of arbitrary elements of M. More formally, the set of all infinite words over Σ is defined by $\Sigma^\omega = \{(a_i)_{i \in \mathbb{N}} | a_i \in \Sigma \text{ for each } i \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of natural numbers. On Σ^ω a *left concatenation* with words from Σ^* is defined. Let $u = b_1 \dots b_k \in \Sigma^*$ with $k \geq 0$ and $b_j \in \Sigma$ for $1 \leq j \leq k$ and $w = (a_i)_{i \in \mathbb{N}} \in \Sigma^\omega$ with $a_i \in \Sigma$ for all $i \in \mathbb{N}$, then $uw = (x_j)_{j \in \mathbb{N}} \in \Sigma^\omega$ with $x_j = b_j$ for $1 \leq j \leq k$ and $x_j = a_{j-k}$ for $k < j$. For $w \in \Sigma^\omega$ the prefix set $\text{pre}(w) \subset \Sigma^*$ is defined by $\text{pre}(w) = \{u \in \Sigma^* |$ it exists $v \in \Sigma^\omega$ with $uv = w\}$. For $L \subset \Sigma^*$ the ω*-language* $L^\omega \subset \Sigma^\omega$ is defined by $L^\omega = \{(a_i)_{i \in \mathbb{N}} \in \Sigma^\omega |$ it exists a strict monotonically increasing function $f : \mathbb{N} \to \mathbb{N}$ with $a_1 \dots a_{f(1)} \in L$ and $a_{f(i)+1} \dots a_{f(i+1)} \in L$ for each $i \in \mathbb{N}\}$. $f : \mathbb{N} \to \mathbb{N}$ is called *strict monotonically increasing* if $f(i) < f(i+1)$ for each $i \in \mathbb{N}$.

Formal languages which describe system behaviour have the characteristic that pre$(u) \subset L$ holds for every word $u \in L$. Such languages are called *prefix closed*. System behaviour is thus described by prefix closed formal languages.

Different formal models of the same system are partially ordered with respect to different levels of abstraction. Formally, abstractions are described by so called alphabetic language homomorphisms. These are mappings $h^* : \Sigma^* \longrightarrow \Sigma'^*$ with $h^*(xy) = h^*(x)h^*(y)$, $h^*(\varepsilon) = \varepsilon$ and $h^*(\Sigma) \subset \Sigma' \cup \{\varepsilon\}$. So they are uniquely defined by corresponding mappings $h : \Sigma \longrightarrow \Sigma' \cup \{\varepsilon\}$. In the following we denote both the mapping h and the homomorphism h^* by h. Inverse homomorphism are denoted by h^{-1}. Let L be a language over the alphabet Σ'. Then $h^{-1}(L)$ is the set of words $w \in \Sigma^*$ such that $h(w) \in L$. In this paper we consider a lot of alphabetic language homomorphisms. So for simplicity we tacitly assume that a mapping between free monoids is an alphabetic language homomorphism if nothing contrary is stated.

To describe a two-sided cooperation, let $\Sigma = \Phi \,\dot\cup\, \Gamma$ where Φ is the set of actions of cooperation partner F and Γ is the set of actions of cooperation partner G. Now a prefix closed language $L \subset (\Phi \,\dot\cup\, \Gamma)^*$ formally defines a two-sided cooperation.

Example 1. *Let* $\Phi = \{\mathrm{f_s}, \mathrm{f_r}\}$ *and* $\Gamma = \{\mathrm{g_r}, \mathrm{g_s}\}$ *and hence* $\Sigma = \{\mathrm{f_s}, \mathrm{f_r}, \mathrm{g_r}, \mathrm{g_s}\}$. *An example for a cooperation* $L \subset \Sigma^*$ *is now given by the automaton in Fig. 1. It describes a simple handshake between* F *(client) and* G *(server), where a client may perform the actions* $\mathrm{f_s}$ *(send a request),* $\mathrm{f_r}$ *(receive a result) and a server may perform the corresponding actions* $\mathrm{g_r}$ *(receive a request) and* $\mathrm{g_s}$ *(send the result).*

Please note that in the following we will denote initial states by a short incoming arrow and final states by double circles. In this automaton all states are final states, since L is prefix closed.

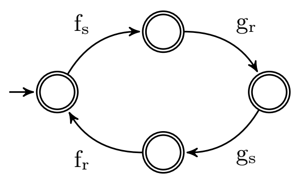


Fig. 1. Automaton for 1-1-cooperation L

For parameter sets I, K and $(i, k) \in I \times K$ let Σ_{ik} denote pairwise disjoint copies of Σ. The elements of Σ_{ik} are denoted by a_{ik} and $\Sigma_{IK} := \dot\bigcup_{(i,k)\in I\times K} \Sigma_{ik}$. The index ik describes the bijection $a \leftrightarrow a_{ik}$ for $a \in \Sigma$ and $a_{ik} \in \Sigma_{ik}$. Now $\mathcal{L}_{IK} \subset \Sigma_{IK}^*$ (prefix-closed) describes a *parameterised system*. To avoid pathological cases we generally assume parameter and index sets to be non empty.

For a cooperation between one partner of type F with two partners of type G in example 1 let

$$\begin{aligned}\Phi_{\{1\}\{1,2\}} =&\{\mathrm{f_{s11}}, \mathrm{f_{r11}}, \mathrm{f_{s12}}, \mathrm{f_{r12}}\},\\ \Gamma_{\{1\}\{1,2\}} =&\{\mathrm{g_{r11}}, \mathrm{g_{s11}}, \mathrm{g_{r12}}, \mathrm{g_{s12}}\} \text{ and}\\ \Sigma_{\{1\}\{1,2\}} =&\Phi_{\{1\}\{1,2\}} \,\dot\cup\, \Gamma_{\{1\}\{1,2\}}.\end{aligned}$$

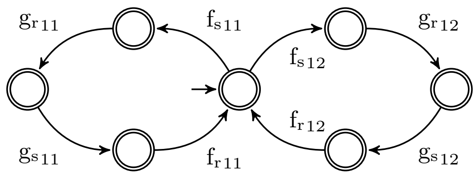


Fig. 2. Automaton for 1-2-cooperation $\mathcal{L}_{\{1\}\{1,2\}}$

A 1-2-cooperation, where each pair of partners cooperates restricted by L and each partner has to finish the handshake it just is involved in before entering a new one, is now given (by reachability analysis) by the automaton in Fig. 2 for $\mathcal{L}_{\{1\}\{1,2\}}$. It shows that one after another client 1 runs a handshake either with server 1 or with server 2. Figure 3 in contrast depicts an automaton for a 2-1-cooperation $\mathcal{L}_{\{1,2\}\{1\}}$ with the same overall number of partners involved but two of type F and one partner of type G. Figure 3 is more complex than Fig. 2 because client 1 and client 2 may start a handshake independently of each other, but server 1 handles these handshakes one after another. A 3-3-cooperation with the same simple behaviour of partners already requires 916 states and 3168 state transitions (computed by the SH verification tool [Ochsenschläger et al. 2000]).

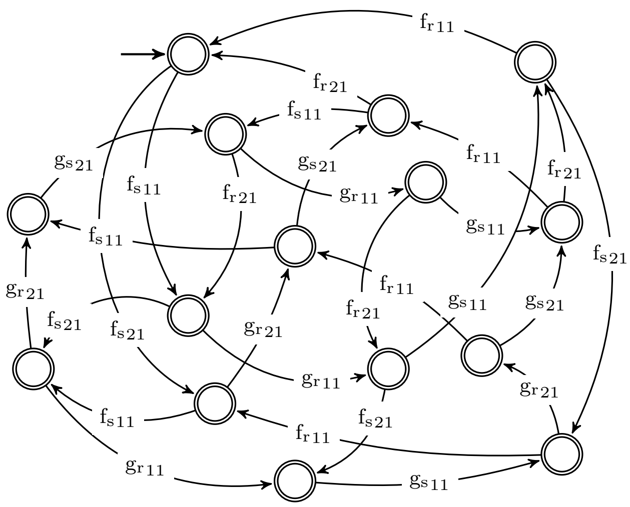


Fig. 3. Automaton for the 2-1-cooperation $\mathcal{L}_{\{1,2\}\{1\}}$

For $(i,k) \in I \times K$, let $\pi_{ik}^{IK} : \Sigma_{IK}^* \to \Sigma^*$ with

$$\pi_{ik}^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Sigma_{ik} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Sigma_{ik} \end{cases} .$$

For *uniformly parameterised systems* $\mathcal{L}_{IK}$ we generally want to have

$$\mathcal{L}_{IK} \subset \bigcap_{(i,k)\in I\times K} ((\pi_{ik}^{IK})^{-1}(L))$$

because from an abstracting point of view, where only the actions of a specific Σ_{ik} are considered, the complex system $\mathcal{L}_{IK}$ is restricted by L.

In addition to this inclusion $\mathcal{L}_{IK}$ is defined by *local schedules* that determine how each "version of a partner" can participate in "different cooperations". More precisely, let $SF \subset \Phi^*$, $SG \subset \Gamma^*$ be prefix closed.

For $(i,k) \in I \times K$, let $\varphi_i^{IK} : \Sigma_{IK}^* \to \Phi^*$ and $\gamma_k^{IK} : \Sigma_{IK}^* \to \Gamma^*$ with

$$\varphi_i^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Phi_{\{i\}K} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Phi_{\{i\}K} \end{cases} \text{ and}$$

$$\gamma_k^{IK}(a_{rs}) = \begin{cases} a \mid a_{rs} \in \Gamma_{I\{k\}} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Gamma_{I\{k\}} \end{cases},$$

where Φ_{IK} and Γ_{IK} are defined correspondingly to Σ_{IK}.

Definition 1 (Uniformly parameterised cooperation $\mathcal{L}_{IK}$). *Let I, K be finite parameter sets, then*

$$\mathcal{L}_{IK} := \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(L) \cap \bigcap_{i\in I} (\varphi_i^{IK})^{-1}(SF) \cap \bigcap_{k\in K} (\gamma_k^{IK})^{-1}(SG)$$

By this definition

$$\mathcal{L}_{\{1\}\{1\}} = (\pi_{11}^{\{1\}\{1\}})^{-1}(L) \cap (\varphi_1^{\{1\}\{1\}})^{-1}(SF) \cap (\gamma_1^{\{1\}\{1\}})^{-1}(SG).$$

As we want $\mathcal{L}_{\{1\}\{1\}}$ being isomorphic to L by the isomorphism

$$\pi_{11}^{\{1\}\{1\}} : \Sigma_{\{1\}\{1\}}^* \to \Sigma^*$$

we additionally need

$$(\pi_{11}^{\{1\}\{1\}})^{-1}(L) \subset (\varphi_1^{\{1\}\{1\}})^{-1}(SF) \text{ and}$$
$$(\pi_{11}^{\{1\}\{1\}})^{-1}(L) \subset (\gamma_1^{\{1\}\{1\}})^{-1}(SG).$$

This is equivalent to $\pi_\Phi(L) \subset SF$ and $\pi_\Gamma(L) \subset SG$, where $\pi_\Phi : \Sigma^* \to \Phi^*$ and $\pi_\Gamma : \Sigma^* \to \Gamma^*$ are defined by

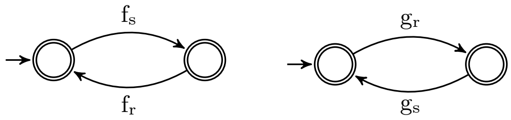
$$\pi_\Phi(a) = \begin{cases} a \mid a \in \Phi \\ \varepsilon \mid a \in \Gamma \end{cases} \text{ and } \pi_\Gamma(a) = \begin{cases} a \mid a \in \Gamma \\ \varepsilon \mid a \in \Phi \end{cases}.$$

So we complete Def. 1 by the additional conditions

$$\pi_\Phi(L) \subset SF \text{ and } \pi_\Gamma(L) \subset SG.$$

Schedules SF and SG that fit to the cooperations given in Example 1 are depicted in Figs. 4(a) and 4(b). Here we have $\pi_\Phi(L) = SF$ and $\pi_\Gamma(L) = SG$.

The system $\mathcal{L}_{IK}$ of cooperations is a typical example of a *complex system.* It consists of several identical components (copies of the two-sided cooperation L), which "interact" in a uniform manner (described by the schedules SF and SG and by the homomorphisms φ_i^{IK} and γ_k^{IK}).



(a) Schedule *SF* (b) Schedule *SG*

Fig. 4. Automata $\mathbb{SF}$ and $\mathbb{SG}$ for the schedules *SF* and *SG*

Remark 1. *It is easy to see that* $\mathcal{L}_{IK}$ *is isomorphic to* $\mathcal{L}_{I'K'}$ *if* I *is isomorphic to* I' *and* K *is isomorphic to* K'*. More precisely, let* $\iota^I_{I'} : I \to I'$ *and* $\iota^K_{K'} : K \to K'$ *be bijections and let* $\iota^{IK}_{I'K'} : \Sigma^*_{IK} \to \Sigma^*_{I'K'}$ *be defined by*

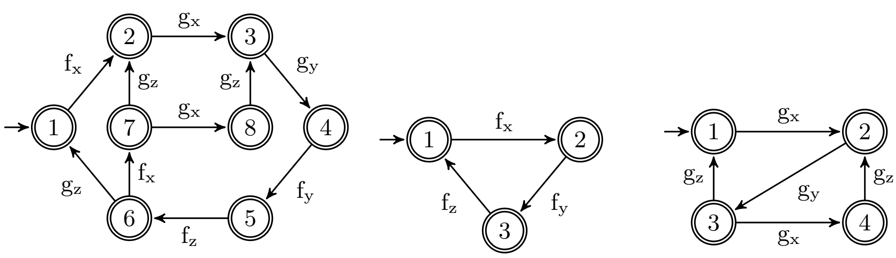
$$\iota^{IK}_{I'K'}(a_{ik}) := a_{\iota^I_{I'}(i)\iota^K_{K'}(k)} \text{ for } a_{ik} \in \Sigma_{IK}.$$

Then $\iota^{IK}_{I'K'}$ *is a isomorphism and* $\iota^{IK}_{I'K'}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'}$*. The set of all these isomorphisms* $\iota^{IK}_{I'K'}$ *defined by corresponding bijections* $\iota^I_{I'}$ *and* $\iota^K_{K'}$ *is denoted by* $\mathcal{I}^{IK}_{I'K'}$*.*

To illustrate the concepts of this paper, we consider the following example.

Example 2. *We consider a system of servers, each of them managing a resource, and clients, which want to use these resources. We assume that as a means to enforce a given* privacy policy *a server has to manage its resource in such a way that no client may access this resource during it is in use by another client (*privacy requirement*). This may be required to ensure* anonymity *in such a way that clients and their actions on a resource cannot be linked by an observer.*

We formalise this system at an abstract level, where a client may perform the actions f_x (send a request), f_y (receive a permission) and f_z (send a free-message), and a server may perform the corresponding actions g_x (receive a request), g_y (send a permission) and g_z (receive a free-message). The possible sequences of actions of a client resp. of a server are given by the automaton $\mathbb{SF}$ resp. $\mathbb{SG}$. The automaton $\mathbb{L}$ describes the 1-1-cooperation of one client and one server (see Fig. 5). These automata define the client-server system $\mathcal{L}_{IK}$.



(a) 1-1-cooperation *L* (b) Schedule *SF* (c) Schedule *SG*

Fig. 5. Automata $\mathbb{L}$, $\mathbb{SF}$ and $\mathbb{SG}$ for Example 2

By *self-similary* [Ochsenschläger and Rieke 2010; 2011] we formalise that for $I' \subset I$ and $K' \subset K$ from an abstracting point of view, where only the actions of

$\Sigma_{I'K'}$ are considered, the complex system $\mathcal{L}_{IK}$ behaves like the smaller subsystem $\mathcal{L}_{I'K'}$. Therefore we now consider special abstractions on $\mathcal{L}_{IK}$.

Definition 2 (Projection abstraction).
For $I' \subset I$ and $K' \subset K$ let $\Pi^{IK}_{I'K'} : \Sigma^*_{IK} \to \Sigma^*_{I'K'}$ with

$$\Pi^{IK}_{I'K'}(a_{rs}) = \begin{cases} a_{rs} \mid a_{rs} \in \Sigma_{I'K'} \\ \varepsilon \mid a_{rs} \in \Sigma_{IK} \setminus \Sigma_{I'K'}. \end{cases}$$

Definition 3 (Self-similarity).
A uniformly parameterised cooperation $\mathcal{L}_{IK}$ is called self-similar *iff*

$$\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'} \text{ for each } I' \times K' \subset I \times K.$$

Self-similarity is a generalisation of $\pi^{IK}_{ik}(\mathcal{L}_{IK}) = L$.

In [Ochsenschläger and Rieke 2010] a sufficient condition for self-similarity is given (see appendix), which is based on deterministic computations in shuffle automata. Under certain regularity restrictions this condition can be verified by a semi-algorithm. In the appendix we show that example 2 is self-similar.

3. UNIFORMLY PARAMETERISED BEHAVIOUR PROPERTIES

Usually behaviour properties of systems are divided into two classes: safety and liveness properties [Alpern and Schneider 1985]. Intuitively a safety property stipulates that “something bad does not happen” and a liveness property stipulates that “something good eventually happens”.

In [Alpern and Schneider 1985] both classes, as well as system behaviour, are formalised in terms of ω-languages, because especially for liveness properties infinite sequences of actions have to be considered.

Definition 4 (linear satisfaction). *According to [Alpern and Schneider 1985], a property E of a system is a subset of Σ^ω. If $S \subset \Sigma^\omega$ represents the behaviour of a system, then S* linearly satisfies *E iff $S \subset E$.*

In [Alpern and Schneider 1985] it is furthermore shown that each property E is the intersection of a safety and a liveness property.

Safety properties $E_s \subset \Sigma^\omega$ are of the form $E_s = \Sigma^\omega \setminus F\Sigma^\omega$ with $F \subset \Sigma^*$, where F is the set of “bad things”.

Liveness properties $E_l \subset \Sigma^\omega$ are characterised by $\text{pre}(E_l) = \Sigma^*$. A typical example of a liveness property is

$$E_l = (\Sigma^* M)^\omega \text{ with } \emptyset \neq M \subset \Sigma^+. \tag{1}$$

This E_l formalises that “always eventually a finite action sequence $m \in M$ happens”.

As we describe system behaviour by prefix closed languages $B \subset \Sigma^*$ we have to “transform” B into an ω-language to apply the framework of [Alpern and Schneider 1985]. This can be done by the Eilenberg-limit $\lim(B)$ [Perrin and Pin 2004].

For prefix closed languages $B \subset \Sigma^*$ their Eilenberg-limit is defined by

$$\lim(B) := \{w \in \Sigma^\omega | \text{pre}(w) \subset B\}.$$

If B contains maximal words u (deadlocks), then these u are not "captured" by $\lim(B)$. Formally the set $\max(B)$ of all maximal words of B is defined by

$$\max(B) := \{u \in B | \text{ if } v \in B \text{ with } u \in \text{pre}(v) \text{ then } v = u\}.$$

Now, using a dummy action #, B can be unambiguously described by

$$\hat{B} := B \cup \max(B)\#^* \subset \hat{\Sigma}^*, \tag{2}$$

where $\# \notin \Sigma$ and $\hat{\Sigma} := \Sigma \dot{\cup} \{\#\}$. By this definition in $\hat{\Sigma}$ the maximal words of B are continued by arbitrary many #'s. So $\hat{B}$ does not contain maximal words. By this construction we now can assume that system behaviour is formalised by prefix closed languages $\hat{B} \subset \Sigma^*\#^* \subset \hat{\Sigma}^*$ without maximal words, and the corresponding infinite system behaviour $S \subset \Sigma^\omega$ is given by $S := \lim(\hat{B})$.

For such an S and safety properties

$$E_s = \hat{\Sigma}^\omega \setminus F\hat{\Sigma}^\omega \text{ with } F \subset \hat{\Sigma}^*$$

it holds

$$S \subset E_s \text{ iff } S \cap F\hat{\Sigma}^\omega = \emptyset \text{ iff } \text{pre}(S) \cap F = \emptyset \text{ iff } \hat{B} \cap F = \emptyset. \tag{3}$$

If $F \subset \Sigma^*$ then $\hat{B} \cap F = \emptyset$ iff $B \cap F = \emptyset$. So

$$S \subset E_s \text{ iff } B \cap F = \emptyset \text{ for } F \subset \Sigma^*. \tag{4}$$

Let $h : \Sigma^* \to \Sigma'^*$ be an alphabetic homomorphism and $F' \subset \Sigma'^*$, then $h(L) \cap F' = \emptyset$ iff $L \cap h^{-1}(F') = \emptyset$. As $h^{-1}(F') \subset \Sigma^*$, (4) implies

$$\lim(\hat{B}) \subset \hat{\Sigma}^\omega \setminus h^{-1}(F')\hat{\Sigma}^\omega \text{ iff } \lim(\widehat{h(B)}) \subset \hat{\Sigma}'^\omega \setminus F'\hat{\Sigma}'^\omega. \tag{5}$$

So by (4) and (5) our approach in [Ochsenschläger and Rieke 2011] is equivalent to the ω-notation of safety properties described by $F \subset \Sigma^*$, and the relation $S \subset E_s$, is compatible with abstractions with respect to such safety properties.

Linear satisfaction (cf. Def. 4) is too strong for systems in our focus with respect to liveness properties, because $S = \lim(\hat{L})$ can contain "unfair" infinite behaviours, which are not elements of E_l.

Let for example $I \supset \{1, 2\}$ and $K \supset \{1\}$ then $\lim(\widehat{\mathcal{L}_{IK}}) \cap \Sigma^\omega_{\{1\}\{1\}} \neq \emptyset$ (infinite action sequences, where only the partners with index 1 cooperate).

If $E_l = \Sigma^*_{IK}\Sigma_{\{2\}\{1\}}\Sigma^\omega_{IK}$ then $\lim(\widehat{\mathcal{L}_{IK}}) \not\subset E_l$.

Instead of neglecting such unfair infinite behaviours in [Nitsche and Ochsenschläger 1996] we defined a weaker satisfaction relation, called *approximate satisfaction*, which implicitly expresses some kind of fairness.

Definition 5 (approximate satisfaction)**.** *A system $S \subset \hat{\Sigma}^\omega$ approximately satisfies a property $E \subset \hat{\Sigma}^\omega$ iff each finite behaviour (finite prefix of an element of S) can be continued to an infinite behaviour, which belongs to E. More formally, $pre(S) \subset pre(S \cap E)$.*

In [Nitsche and Ochsenschläger 1996] it is shown, that

for safety properties linear satisfaction and
approximate satisfaction are equivalent. (6)

With respect to approximate satisfaction liveness properties stipulate that "something good" eventually is possible.

Concerning properties E not of the form $E = \hat{\Sigma}^\omega \setminus F\hat{\Sigma}^\omega$ with $F \subset \Sigma^*$ approximate satisfaction is not compatible with abstractions in such sense, that there exist pairs of concrete and abstract systems related by homomorphisms such that the abstract system approximately satisfies such a property E but the concrete system does not approximately satisfy a "corresponding" property. In [Ochsenschläger 1992] and [Nitsche and Ochsenschläger 1996] such examples are discussed and a property of abstractions is given that overcomes this problem. This property is called *simplicity* of an alphabetic homomorphism $h : \Sigma^* \to \Sigma'^*$ with respect to a prefix closed language $B \subset \Sigma^*$ and it is formalised in terms of continuation possibilities in B and $h(B)$.

Definition 6. *An alphabetic language homomorphism* $h : \Sigma^* \to \Sigma'^*$ *is* simple *on* $B \subset \Sigma^*$ *iff for each* $w \in B$ *there exists* $u \in h(w)^{-1}(h(B))$ *such that* $u^{-1}(h(w^{-1})(B))) = u^{-1}(h(w)^{-1}(h(B)))$.

In [Ochsenschläger 1992] some sufficient conditions for simplicity are given. For our purpose the following is helpful (for the proof cf. the appendix)

Theorem 1. *If for each* $y \in B$ *there exists* $z \in y^{-1}(B)$ *with* $h((yz)^{-1}(B)) = (h(yz))^{-1}(h(B))$ *then* h *is simple on* B.

To formulate the implication of simplicity we have to "extend" h to $\hat{\Sigma}^\omega$.

Let $\hat{h} : \hat{\Sigma}^* \to \widehat{\Sigma'}^*$ be the homomorphisms defined by $\hat{h}(a) := h(a)$ for $a \in \Sigma$ and $\hat{h}(\#) := \#$.

$$\begin{aligned} &\text{For } x \in \hat{\Sigma}^\omega \text{ either } \lim(\hat{h}(\text{pre}(x))) = \{y\} \subset \widehat{\Sigma'}^\omega \\ &\text{or } \max(\hat{h}(\text{pre}(x))) = \{z\} \subset \Sigma'^*. \end{aligned} \tag{7}$$

Now let $\hat{h}_\omega : \hat{\Sigma}^\omega \to \widehat{\Sigma'}^\omega$ be defined for $x \in \hat{\Sigma}^\omega$ by $\hat{h}_\omega(x) := y$ if $\lim(\hat{h}(\text{pre}(x))) = \{y\} \subset \widehat{\Sigma'}^\omega$ and $\hat{h}_\omega(x) := z\{\#\}^\omega$ if $\max(\hat{h}(\text{pre}(x))) = \{z\} \subset \Sigma'^*$.

$\hat{h}_\omega$ is not an homomorphism but it has the following properties:
If $w = uv \in \hat{\Sigma}^\omega$ with $u \in \hat{\Sigma}^*$ and $v \in \hat{\Sigma}^\omega$ then

$$\hat{h}_\omega = \hat{h}(u)\hat{h}_\omega(v). \tag{8}$$

If $w' = u'a'v' \in \widehat{\Sigma'}^\omega$ with $u' \in \widehat{\Sigma'}^*$, $a' \in \Sigma'$, $v' \in \widehat{\Sigma'}^\omega$ and $w \in \hat{\Sigma}^\omega$ with $\hat{h}_\omega(w) = w'$ then

$$\begin{aligned} &w = uav \text{ with } u \in \hat{\Sigma}^*, a \in \Sigma, v \in \hat{\Sigma}^\omega, \\ &\hat{h}(u) = u', h(a) = a' \text{ and } \hat{h}_\omega(v) = v'. \end{aligned} \tag{9}$$

In [Nitsche and Ochsenschläger 1996] the following has been proven:

Theorem 2. *If* h *is simple on a regular prefix closed language* B *then*

$$\begin{aligned} &pre(lim(\widehat{h(B)})) \subset pre(lim(\widehat{h(B)}) \cap E') \text{ implies} \\ &pre(lim(\hat{B})) \subset pre(lim(\hat{B}) \cap \hat{h}_\omega^{-1}(E')) \end{aligned}$$

for each $E' \subset \widehat{\Sigma'}^{\omega}$.

Here $\hat{h}_{\omega}^{-1}(E')$, which is approximately satisfied by the concrete system $\lim(\hat{B})$, is the corresponding property to E', which is approximately satisfied by the abstract system $\lim(\widehat{h(B)})$. It has been proven that simplicity of h on B is necessary for the set of implications in theorem 2.

In [Ochsenschläger and Rieke 2011] safety properties are formalised by formal languages $F \subset \Sigma^*$ and it is defined that a prefix closed language $B \subset \Sigma^*$ satisfies such a safety property F iff $L \cap F = \emptyset$. By (4) and (6) this is equivalent to the statement that $\lim(\hat{L})$ approximately satisfies the safety property

$$\hat{\Sigma}^{\omega} \setminus F\hat{\Sigma}^{\omega}. \tag{10}$$

In [Ochsenschläger and Rieke 2011] uniformly parameterised safety properties are generated by safety properties $\mathring{F} \subset \Sigma^*_{\mathring{I}\mathring{K}}$ and defined in such a way that a parameterised system $\mathcal{L}_{IK} \subset \Sigma^*_{IK}$ satisfies the generated parameterised safety property iff $\mathcal{L}_{IK}$ satisfies each safety property $(\Pi^{IK}_{I'K'})^{-1}(\iota^{\mathring{I}\mathring{K}}_{I'K'}(\mathring{F}))$ with $I' \subset I$, $K' \subset K$ and $\iota^{\mathring{I}\mathring{K}}_{I'K'} \in \mathcal{I}^{\mathring{I}\mathring{K}}_{I'K'}$, where $\mathcal{I}^{\mathring{I}\mathring{K}}_{I'K'}$ is the set of all isomorphisms $\iota^{\mathring{I}\mathring{K}}_{I'K'} : \Sigma^*_{\mathring{I}\mathring{K}} \to \Sigma^*_{I'K'}$ generated by bijections $\iota^{\mathring{I}}_{I'} : \mathring{I} \to I'$ and $\iota^{\mathring{K}}_{K'} : \mathring{K} \to K'$ in such a way that

$$\iota^{\mathring{I}\mathring{K}}_{I'K'}(a_{ik}) := a_{\iota^{\mathring{I}}_{I'}(i)\iota^{\mathring{K}}_{K'}(k)} \tag{11}$$

for $a_{ik} \in \Sigma_{\mathring{I}\mathring{K}}$. We now want to generalise this idea to arbitrary system properties formulated as subsets of $\hat{\Sigma}^{\omega}$. First of all we notice that for index sets $\mathring{I}$, I', $\mathring{K}$ and K' each bijection $\iota^{\mathring{I}}_{I'} : \mathring{I} \to I'$ and $\iota^{\mathring{K}}_{K'} : \mathring{K} \to K'$ generates an isomorphism $\hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'} : \hat{\Sigma}^*_{\mathring{I}\mathring{K}} \to \hat{\Sigma}^*_{I'K'}$ by $\hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'}(a) := \iota^{\mathring{I}\mathring{K}}_{I'K'}(a)$ for $a \in \Sigma_{\mathring{I}\mathring{K}}$ and $\hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'}(\#) := \#$.
For each $w \in \hat{\Sigma}^{\omega}_{\mathring{I}\mathring{K}}$ $\lim(\hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'}(\text{pre}(w))) = \{w'\} \in \hat{\Sigma}^{\omega}_{I'K'}$.
Now the mapping $\widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'} : \hat{\Sigma}^{\omega}_{\mathring{I}\mathring{K}} \to \hat{\Sigma}^{\omega}_{I'K'}$ defined for each $w \in \hat{\Sigma}^{\omega}_{\mathring{I}\mathring{K}}$ by

$$\widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'}(w) := w' \text{ with } \lim(\hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'}(\text{pre}(w))) = \{w'\},$$

is a bijection. The set of all these bijections $\widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'}$ we denote by $\widehat{\mathcal{I}\omega}^{\mathring{I}\mathring{K}}_{I'K'}$.

$\widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'}$ is "like an isomorphism" because for each $w \in \hat{\Sigma}^{\omega}_{\mathring{I}\mathring{K}}$ holds:

$$w = uv \text{ with } u \in \hat{\Sigma}^*_{\mathring{I}\mathring{K}} \text{ and } v \in \hat{\Sigma}^{\omega}_{\mathring{I}\mathring{K}} \text{iff } \widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'}(w) = \hat{\iota}^{\mathring{I}\mathring{K}}_{I'K'}(u)\widehat{\iota\omega}^{\mathring{I}\mathring{K}}_{I'K'}(v). \tag{12}$$

For finite index sets $\mathring{I}$, I, $\mathring{K}$ and K let

$$\hat{\mathcal{I}}[(\mathring{I},\mathring{K}),(I,K)] := \bigcup_{I' \subset I, K' \subset K} \widehat{\mathcal{I}\omega}^{\mathring{I}\mathring{K}}_{I'K'}.$$

Note that

$$\hat{\mathcal{I}}[(\mathring{I},\mathring{K}),(I,K)] = \emptyset \text{ if } |\mathring{I}| > |I| \text{ or } |\mathring{K}| > |K|, \tag{13}$$

where $|I|$ denotes the cardinality of the set I.

Now let $\mathring{E} \subset \widehat{\Sigma_{\mathring{I}\mathring{K}}}^{\omega}$, with fixed index sets $\mathring{I}$ and $\mathring{K}$, be an arbitrary property.

Motivated by theorem 2 and [Ochsenschläger and Rieke 2011] for finite index sets I and K we define

$$\mathcal{E}_{IK}^{\mathring{E}} := [((\widehat{\Pi_{I'K'}^{IK}})_\omega)^{-1}(\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E}))]_{\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}} \in \hat{\mathcal{I}}[(\mathring{I},\mathring{K}),(I,K)]}. \tag{14}$$

We say that

$\lim(\widehat{\mathcal{L}_{IK}})$ *approximately satisfies such a family* $\mathcal{E}_{IK}^{\mathring{E}}$ *of properties* iff
$\lim(\widehat{\mathcal{L}_{IK}})$ approximately satisfies each of the properties
$((\widehat{\Pi_{I'K'}^{IK}})_\omega)^{-1}(\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E}))$ for $\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}} \in \hat{\mathcal{I}}[(\mathring{I},\mathring{K}),(I,K)]$. (15)

On account of (13) it makes sense to consider finite families of $\mathcal{E}_{IK}^{\mathring{E}}$.

Definition 7 (uniformly parameterised behaviour property)**.**
Let T, I and K be finite index sets. For each $t \in T$ let $\mathring{E}_t \subset \widehat{\Sigma_{\mathring{I}_t\mathring{K}_t}}^\omega$ and $\mathcal{E}_{IK}^{\mathring{E}_t}$ be defined as in (14)*. Then $\mathcal{E}_{IK} := (\mathcal{E}_{IK}^{\mathring{E}_t})_{t\in T}$ is called a* uniformly parameterised behaviour property.

We say that lim$(\widehat{\mathcal{L}_{IK}})$ approximately satisfies $\mathcal{E}_{IK}$ *iff lim$(\widehat{\mathcal{L}_{IK}})$ approximately satisfies each $\mathcal{E}_{IK}^{\mathring{E}_t}$ for $t \in T$ as defined in* (15).

If $\mathring{E} = \widehat{\Sigma_{\mathring{I}\mathring{K}}}^\omega \setminus \mathring{F}\widehat{\Sigma_{\mathring{I}\mathring{K}}}^\omega$ with $\mathring{F} \subset \Sigma_{\mathring{I}\mathring{K}}^*$ then by (12)

$$\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E}) = \widehat{\Sigma_{I'K'}}^\omega \setminus \iota_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{F})\widehat{\Sigma_{I'K'}}^\omega$$

and by (8) and (9)

$$((\widehat{\Pi_{I'K'}^{IK}})_\omega)^{-1}(\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E})) = \widehat{\Sigma_{IK}}^\omega \setminus (\Pi_{I'K'}^{IK})^{-1}(\iota_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{F}))\widehat{\Sigma_{IK}}^\omega.$$

Now (10) and (11) imply that definition 7 generalises the corresponding definitions of [Ochsenschläger and Rieke 2011].

If $\Pi_{I'K'}^{IK}$ is simple on a regular $\mathcal{L}_{IK}$ for $I' \subset I$ and $K' \subset K$ and if $\mathring{E} \subset \widehat{\Sigma_{\mathring{I}\mathring{K}}}^\omega$ is an arbitrary property, then by theorem 2 $\lim(\widehat{\mathcal{L}_{IK}})$ approximately satisfies $((\widehat{\Pi_{I'K'}^{IK}})_\omega)^{-1}(\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E}))$ if $\lim(\widehat{\Pi_{I'K'}^{IK}(\mathcal{L}_{IK})})$ approximately satisfies $\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E})$. If $\mathcal{L}_{IK}$ is self-similar, then $\Pi_{I'K'}^{IK}(\mathcal{L}_{IK}) = \mathcal{L}_{I'K'}$ for each $I' \subset I$ and $K' \subset K$. If $\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}} \in \widehat{\mathcal{I}\omega}_{I'K'}^{\mathring{I}\mathring{K}}$ then by (12) $\lim(\widehat{\mathcal{L}_{I'K'}})$ approximately satisfies $\widehat{\iota\omega}_{I'K'}^{\mathring{I}\mathring{K}}(\mathring{E})$ iff $\lim(\widehat{\mathcal{L}_{\mathring{I}\mathring{K}}})$ approximately satisfies $\mathring{E}$. So we get

Theorem 3. *Let I, K, $\mathring{I}$ and $\mathring{K}$ be finite index sets with $|\mathring{I}| \leq |I|$ and $|\mathring{K}| \leq |K|$. Let $\mathcal{L}_{IK}$ be a uniformly parameterised, self-similar regular system of cooperations and let $\Pi_{I'K'}^{IK}$ simple on $\mathcal{L}_{IK}$ for each $I' \subset I$ and $K' \subset K$ with $|\mathring{I}| = |I'|$ and $|\mathring{K}| = |K'|$. Then for $\mathring{E} \subset \widehat{\Sigma_{\mathring{I}\mathring{K}}}^\omega$ lim$(\widehat{\mathcal{L}_{IK}})$ approximately satisfies $\mathcal{E}_{IK}^{\mathring{E}}$ if lim$(\widehat{\mathcal{L}_{\mathring{I}\mathring{K}}})$ approximately satisfies $\mathring{E}$.*

Remark 2. *By the well known closure properties of the family of regular languages [Sakarovitch 2009] $\mathcal{L}_{IK}$ is regular if it is defined by regular languages L, SF, SG.*

Self-similarity of $\mathcal{L}_{IK}$ is given by sufficient conditions in [Ochsenschläger and Rieke 2010] (see appendix).

If $\mathcal{L}_{IK}$ is regular and $\mathring{E}$ ω-regular [Perrin and Pin 2004] then approximately satisfaction of $\mathring{E}$ by lim($\widehat{\mathcal{L}_{\mathring{I}\mathring{K}}}$) can be checked by finite state methods [Perrin and Pin 2004] (intersection of ω-regular languages).

Many practical liveness properties are of the form (1). Let us consider a prefix closed language $B \subset \Sigma^*$ and a formal language $\emptyset \neq M \subset \Sigma^+$. By definition 5 $\lim(\hat{B})$ approximately satisfies $(\hat{\Sigma}^* M)^\omega$ iff each $u \in B$ is prefix of $v \in B$ with

$$v^{-1}(B) \cap M \neq \emptyset. \tag{16}$$

If B and M are regular sets, then (16) can be checked by usual automata algorithms [Sakarovitch 2009] without referring to $\lim(\hat{B}) \cap (\hat{\Sigma}^* M)^\omega$.

If $h : \Sigma^* \to \Sigma'^*$ is an alphabetic homomorphism and $M' \subset \Sigma^+$, then by (8) and (9)

$$\hat{h}_\omega^{-1}((\hat{\Sigma'}^* M')^\omega) = (\hat{\Sigma}^* h^{-1}(M'))^\omega \subset \hat{\Sigma}^\omega, \tag{17}$$

which is also of the form (1).

Let us now consider the prefix closed language $L \subset \Sigma^*$ of example 2 and the "phase" $P \subset \Sigma^+$ given by the automaton $\mathbb{P}$ in Fig. 6.

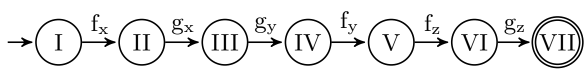


Fig. 6. Automaton $\mathbb{P}$

$\lim(\hat{L})$ approximately satisfies the liveness property $(\hat{\Sigma}^* P)^\omega \subset \hat{\Sigma}^*$,
because the automaton $\mathbb{L}$ in Fig. 5(a) is strongly connected and $P \subset L$. (18)

(18) states that in the 1-1-cooperation $\lim(\hat{L})$ always eventually a "complete run through the phase P" is possible.

Let now

$$\mathring{P} := (\pi_{11}^{\{1\}\{1\}})^{-1} P \subset \Sigma^+_{\{1\}\{1\}} \text{ and}$$
$$\mathring{E} := (\widehat{\Sigma_{\{1\}\{1\}}}^{*} \mathring{P})^\omega \subset \widehat{\Sigma_{\{1\}\{1\}}}^{\omega}. \tag{19}$$

As $\pi_{11}^{\{1\}\{1\}} : \Sigma^*_{\{1\}\{1\}} \to \Sigma^*$ is an isomorphism then by (18) $\lim(\widehat{\mathcal{L}_{\{1\}\{1\}}})$ approximately satisfies $\mathring{E}$.

By remark 2 $\mathcal{L}_{IK}$ is regular in example 2, and in the appendix it is shown that $\mathcal{L}_{IK}$ is self-similar. So if we prove simplicity of $\Pi^{IK}_{I'K'}$ on $\mathcal{L}_{IK}$, which will be done in section 5 then by theorem 3

$$\lim(\widehat{\mathcal{L}_{IK}}) \text{ approximately satisfies } \mathcal{E}^{\mathring{E}}_{IK} \tag{20}$$

for each finite indes set I and K.

By (17) (20) states that for each pair of clients and servers always eventually a "complete run through a phase P" is possible w.r.t. the abstraction, where only the actions of this client and server are considered.

4. COOPERATIONS BASED ON PHASES

The schedule SG of example 2 shows that a server may cooperate with two clients partly in an interleaving manner. To formally capture such behaviour, in [Ochsenschläger and Rieke 2010] cooperations are structured into phases. This formalism is based on iterated shuffle-products [Jantzen 1985] and leads in section 5 to sufficient conditions for simplicity of $\Pi^{IK}_{I'K'}$ on $\mathcal{L}_{IK}$.

Definition 8.

$$P^{\sqcup\!\sqcup} := \Theta^{\mathbb{N}}[\bigcap_{t\in\mathbb{N}}(\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})] \textit{ for } P \subset \Sigma^*.$$

For the definition of the homomorphisms $\Theta^{\mathbb{N}}$ and $\tau_t^{\mathbb{N}}$, let $t \in \mathbb{N}$, and for each t let Σ_t be a copy of Σ. Let all Σ_t be pairwise disjoint. The index t describes the bijection $a \leftrightarrow a_t$ for $a \in \Sigma$ and $a_t \in \Sigma_t$.

Let $\Sigma_{\mathbb{N}} := \bigcup_{t\in\mathbb{N}} \Sigma_t$, and for each $t \in \mathbb{N}$ let the homomorphisms $\tau_t^{\mathbb{N}}$ and $\Theta^{\mathbb{N}}$ be defined by

$$\tau_t^{\mathbb{N}} : \Sigma_{\mathbb{N}}^* \to \Sigma^* \text{ with } \tau_t^{\mathbb{N}}(a_s) = \begin{cases} a \mid & a_s \in \Sigma_t \\ \varepsilon \mid & a_s \in \Sigma_{\mathbb{N}} \setminus \Sigma_t \end{cases}$$

and $\Theta^{\mathbb{N}} : \Sigma_{\mathbb{N}}^* \to \Sigma^*$ with $\Theta^{\mathbb{N}}(a_t) := a$ for $a_t \in \Sigma_t$ and $t \in \mathbb{N}$.

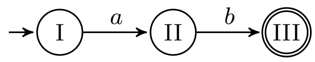


Fig. 7. Automaton $\mathbb{P}$ for P= {ab}

Let for example $P = \{ab\}$ be given by the Automaton $\mathbb{P}$ in Fig. 7. Then $aabb \in P^{\sqcup\!\sqcup}$ because $aabb = \Theta^{\mathbb{N}}(a_1a_2b_2b_1)$ and $\tau_1^{\mathbb{N}}(a_1a_2b_2b_1) = \tau_2^{\mathbb{N}}(a_1a_2b_2b_1) = ab \in P$ and $\tau_t^{\mathbb{N}}(a_1a_2b_2b_1) = \varepsilon$ for $t \in \mathbb{N} \setminus \{1, 2\}$.

$a_1a_2b_2b_1$ is a structured representation of $aabb$ (see definition 10).

Definition 8 looks different to the usual one of iterated shuffle products, as for example in [Jantzen 1985]. But it is easy to see that they are equivalent. We use our kind of definition, as it is more adequate to the considerations in this paper.

Definition 9.
A prefix closed language $B \subset \Sigma^*$ is based on a phase $P \subset \Sigma^*$, *iff* $B = \text{pre}(P^{\sqcup\!\sqcup} \cap B)$.

If B is based on P, then $B \subset \text{pre}(P^{\sqcup\!\sqcup}) = (\text{pre}(P))^{\sqcup\!\sqcup}$ and
$B = \text{pre}(P))^{\sqcup\!\sqcup} \cap B$.

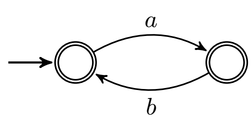


Fig. 8. Automaton $\mathbb{B}$ for B

Let for example $P = \{ab\}$ and B be given by the automaton $\mathbb{B}$ in Fig. 8. Then $P^{\sqcup\!\sqcup} \cap B = \{ab\}^*$. This implies that B is based on P.

Generally each B is based on infinitely many phases.

– If B is based on P then B is based on P' for each $P' \supset P$.
– Each $B \subset \Sigma^*$ is based on Σ because $\Sigma^{\sqcup\!\sqcup} = \Sigma^*$.

The appropriate phases for our purposes will be discussed in Sect. 5.

For the subsequent lemmata, which are proven in [Ochsenschläger and Rieke 2010] and will be used in Sect. 5, let S and T be arbitrary index sets and $M \subset \Sigma^*$.

Let $\tau_t^T : \Sigma_T^* \to \Sigma^*$ for $t \in T$ and $\Theta^T : \Sigma_T^* \to \Sigma^*$ be defined like $\tau_t^{\mathbb{N}}$ and $\Theta^{\mathbb{N}}$.

For each $S' \subset S$ and $T' \subset T$ let

$\Theta_{S'}^{S' \times T'} : \Sigma_{S' \times T'}^* \to \Sigma_{S'}^*$ with $\Theta_{S'}^{S' \times T'}(a_{(s,t)}) := a_s$ for each $a_{(s,t)} \in \Sigma_{S' \times T'}$ and

$\Theta_{T'}^{S' \times T'} : \Sigma_{S' \times T'}^* \to \Sigma_{T'}^*$ with $\Theta_{T'}^{S' \times T'}(a_{(s,t)}) := a_t$ for each $a_{(s,t)} \in \Sigma_{S' \times T'}$.

Lemma 1 (Shuffle-lemma 1).
Let S, T arbitrary index sets and $M \subset \Sigma^*$, then

$$\bigcap_{s \in S} (\tau_s^S)^{-1}[\Theta^T(\bigcap_{t \in T} (\tau_t^T)^{-1}(M))] = \Theta_S^{S \times T}[\bigcap_{(s,t) \in S \times T} (\tau_{(s,t)}^{S \times T})^{-1}(M)], \tag{21a}$$

and, since $\Theta^{S \times T} = \Theta^S \circ \Theta_S^{S \times T}$,

$$\Theta^S[\bigcap_{s \in S} (\tau_s^S)^{-1}[\Theta^T(\bigcap_{t \in T} (\tau_t^T)^{-1}(M))]] = \Theta^{S \times T}[\bigcap_{(s,t) \in S \times T} (\tau_{(s,t)}^{S \times T})^{-1}(M)]. \tag{21b}$$

Definition 10. *Let S be an arbitrary index set. For each $x \in \Theta^S[\bigcap_{s \in S} (\tau_s^S)^{-1}(M)]$ there exists $u \in \bigcap_{s \in S} (\tau_s^S)^{-1}(M)$ such that $x = \Theta^S(u)$. We call u a* structured representation of x w.r.t. S. *For $x \in \Sigma^*$ let $SR_M^S(x) := (\Theta^S)^{-1}(x) \cap [\bigcap_{s \in S} (\tau_s^S)^{-1}(M)]$. It is the set of all structured representations of x w.r.t. S and fixed $M \subset \Sigma^*$.*

Now $x \in P^{\sqcup\!\sqcup}$ iff there exists a countable index set S with $SR_{(P \cup \{\epsilon\})}^S(x) \neq \emptyset$ (see Lemma 2). If $x \in P^{\sqcup\!\sqcup}$, then generally $SR_{(P \cup \{\epsilon\})}^S(x)$ contains more than one element.

Lemma 2 (Shuffle-lemma 2).
If a bijection between S and T exists, then $\Theta^S[\bigcap_{s \in S} (\tau_s^S)^{-1}(M)] = \Theta^T[\bigcap_{t \in T} (\tau_t^T)^{-1}(M)]$ for $M \subset \Sigma^*$.

For an arbitrary index set S and $S' \subset S$ let

$$\Pi_{S'}^S : \Sigma_S^* \to \Sigma_{S'}^* \quad \text{with } \Pi_{S'}^S(a_s) = \begin{cases} a_s \mid a_s \in \Sigma_{S'} \\ \varepsilon \mid a_s \in \Sigma_S \setminus \Sigma_{S'} \end{cases}.$$

Lemma 3 (Shuffle-lemma 3).
Let $M \subset \Sigma^*$, S, T index sets and $y \in \Sigma_{S \times T}^*$ with $\tau_{(s,t)}^{S \times T}(y) \in M$ for each $(s,t) \in S \times T$ and $x = \Theta_S^{S \times T}(y) \in \Sigma_S^*$, then $\Pi_{S' \times T}^{S \times T}(y) \in SR_M^{S' \times T}(\Theta^{S'}(\Pi_{S'}^S(x)))$ for each $S' \subset S$.

Remark 3. *The hypotheses of this lemma are given by* (21a).

In [Ochsenschläger and Rieke 2010] an automaton representation $\mathbb{A}^{\sqcup\!\sqcup}$ for $P^{\sqcup\!\sqcup}$ is given, which will illustrate "how a language B is based on a phase P".

Let $P \subset \Sigma^*$ and $\mathbb{A} = (\Sigma, Q, \Delta, q_0, F)$ with $\Delta \subset Q \times \Sigma \times Q$, $q_0 \in Q$ and $F \subset Q$ be an (not necessarily finite) automaton that accepts P. To exclude pathological cases we assume $\varepsilon \notin P \neq \emptyset$. A consequence of this is in particular that $q_0 \notin F$.

For the construction of $\mathbb{A}^{\sqcup\!\sqcup}$ the set $\mathbb{N}_0^Q$ (set of all functions from Q in $\mathbb{N}_0$) plays a central role. In $\mathbb{N}_0^Q$ we distinguish the following functions:

$$0 \in \mathbb{N}_0^Q \text{ with } 0(x) = 0 \text{ for each } x \in Q, \text{ and for } q \in Q \text{ the function}$$
$$1_q \in \mathbb{N}_0^Q \text{ with } 1_q(x) = \begin{cases} 1 \mid x = q \\ 0 \mid x \in Q \setminus \{q\} \end{cases}.$$

As usual for numerical functions, a partial order as well as addition and partial subtraction are defined:

For $f, g \in \mathbb{N}_0^Q$ let

$$f \geqslant g \text{ iff } f(x) \geqslant g(x) \text{ for each } x \in Q,$$
$$f + g \in \mathbb{N}_0^Q \text{ with } (f+g)(x) := f(x) + g(x) \text{ for each } x \in Q, \text{ and}$$
$$\text{for } f \geqslant g, f - g \in \mathbb{N}_0^Q \text{ with } (f-g)(x) := f(x) - g(x) \text{ for each } x \in Q.$$

The key idea of $\mathbb{A}^{\sqcup\!\sqcup}$ is, to record in the functions of $\mathbb{N}_0^Q$ how many "open phases" are in each state $q \in Q$ respectively. Its state transition relation $\Delta^{\sqcup\!\sqcup}$ is composed of four subsets whose elements describe

– the "entry into a new phase",
– the "transition within an open phase",
– the "completion of an open phase",
– the "entry into a new phase with simultaneous completion of this phase".

With these definitions we now define the *shuffle automaton* $\mathbb{A}^{\sqcup\!\sqcup}$ as follows:

Definition 11 (shuffle automaton).
The shuffle automaton $\mathbb{A}^{\sqcup\!\sqcup} = (\Sigma, \mathbb{N}_0^Q, \Delta^{\sqcup\!\sqcup}, 0, \{0\})$ *w.r.t.* $\mathbb{A}$ *is an automaton with infinite state set* $\mathbb{N}_0^Q$*, the initial state* 0*, which is the only final state and*

$$\begin{aligned}
\Delta^{\sqcup\!\sqcup} :=& \{(f, a, f + 1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \\
& \quad (q_0, a, p) \in \Delta \text{ and it exists } (p, x, y) \in \Delta\} \cup \\
& \{(f, a, f + 1_p - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \\
& \quad f \geqslant 1_q, (q, a, p) \in \Delta \text{ and it exists } (p, x, y) \in \Delta\} \cup \\
& \{(f, a, f - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \\
& \quad f \geqslant 1_q, (q, a, p) \in \Delta \text{ and } p \in F\} \cup \\
& \{(f, a, f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid (q_0, a, p) \in \Delta \text{ and } p \in F\}.
\end{aligned}$$

Accepting of a word $w \in \Sigma^*$ *is defined in the usual manner [Sakarovitch 2009].*

Generally $\mathbb{A}^{\sqcup\!\sqcup}$ is a non-deterministic automaton with an infinite state set. In the literature such automata are called multicounter automata [Björklund and Bojanczyk 2007] and it is known that they accept the iterated shuffle products [Jedrzejowicz 1999]. For our purposes deterministic computations of these automata are very important. To analyse these aspects more deeply we use our own notation and proof of the main theorems.

Let for example $P = \{ab\}$ (cf. Fig. 7). Then the states $f : Q \to \mathbb{N}_0$ of the automaton $\mathbb{P}^{\sqcup\!\sqcup}$ are described by the sets $\{(q, n) \in Q \times \mathbb{N}_0 | f(q) = n \neq 0\}$.

$$\emptyset \xrightarrow{a} \{(\mathrm{II}, 1)\} \xrightarrow{a} \{(\mathrm{II}, 2)\} \xrightarrow{b} \{(\mathrm{II}, 1)\} \xrightarrow{b} \emptyset$$

is the only computation of $aabb \in P^{\sqcup\!\sqcup}$ in $\mathbb{P}^{\sqcup\!\sqcup}$; it is an accepting computation.

Example 3. *Let $P' = \{ab, aab, b\}$.*

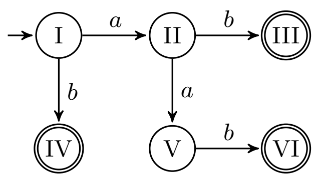


Fig. 9. Automaton $\mathbb{P}'$ for P'

There are three accepting computations of $aabb \in P'^{\sqcup\!\sqcup}$ in $\mathbb{P}'^{\sqcup\!\sqcup}$:

$$\emptyset \xrightarrow{a} \{(II, 1)\} \xrightarrow{a} \{(II, 2)\} \xrightarrow{b} \{(II, 1)\} \xrightarrow{b} \emptyset$$
$$\emptyset \xrightarrow{a} \{(II, 1)\} \xrightarrow{a} \{(V, 1)\} \xrightarrow{b} \emptyset \xrightarrow{b} \emptyset$$
$$\emptyset \xrightarrow{a} \{(II, 1)\} \xrightarrow{a} \{(V, 1)\} \xrightarrow{b} \{(V, 1)\} \xrightarrow{b} \emptyset$$

and four not accepting computations, e.g.

$$\emptyset \xrightarrow{a} \{(II, 1)\} \xrightarrow{a} \{(V, 1)\} \xrightarrow{b} \{(V, 1)\} \xrightarrow{b} \{(V, 1)\}.$$

In [Ochsenschläger and Rieke 2010] it is shown that $\mathbb{A}^{\sqcup\!\sqcup}$ accepts $P^{\sqcup\!\sqcup}$.

Let $P \subset \Sigma^+$ be defined by the automaton $\mathbb{P}$ in Fig. 6 and let $L \subset \Sigma^*$ be defined by the automaton $\mathbb{L}$ in Fig. 5(a). $L \cap P^{\sqcup\!\sqcup}$ is accepted by the following product automaton [Sakarovitch 2009] of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$ (see Fig. 10), where the states $f : Q_P \to \mathbb{N}_0$ of the automaton $\mathbb{P}^{\sqcup\!\sqcup}$ are described by the sets $\{(q, n) \in Q_P \times \mathbb{N}_0 \mid f(q) = n \neq 0\}$ and $Q_P = \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}, \mathrm{VII}\}$.

As this automaton is strongly connected and isomorphic to $\mathbb{L}$ (without considering final states), L is based on phase P.

The states $(7, \{(\mathrm{VI}, 1), (\mathrm{II}, 1)\})$ and $(8, \{(\mathrm{VI}, 1), (\mathrm{III}, 1)\})$ show that L is "in this states involved in two phases".

Note that this product automaton is finite and deterministic.

As deterministic computations in $\mathbb{A}^{\sqcup\!\sqcup}$ play an important role (see theorem 4) for simplicity we assume that $\mathbb{A}$ is deterministic. I.e., the state transition relation Δ can be described by a partial function $\delta : Q \times \Sigma \to Q$ which is extended to a partial function $\delta : Q \times \Sigma^* \to Q$ as usual [Sakarovitch 2009]. Additionally we assume

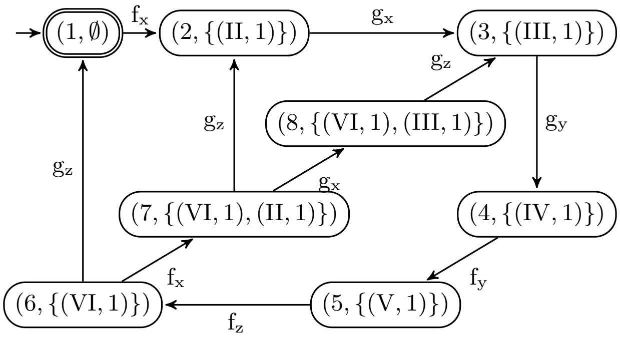


Fig. 10. Product automaton of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$

that $\mathbb{A}$ does not contain superfluous states, i.e. $\delta(q_0, \mathrm{pre}(P)) = Q$. So $\Delta^{\sqcup\!\sqcup}$ can be represented by

$$\Delta^{\sqcup\!\sqcup} = \tilde{\Delta} \cup \mathring{\Delta} \cup \bar{\Delta} \cup \tilde{\bar{\Delta}} \text{ with}$$

$$\tilde{\Delta} = \{(f, a, f + 1_p) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0, a) = p \text{ and it exists } b \in \Sigma \text{ such that } \delta(p, b) \text{ is defined}\},$$

$$\mathring{\Delta} = \{(f, a, f + 1_p - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q, \delta(q, a) = p \text{ and it exists } b \in \Sigma \text{ such that } \delta(p, b) \text{ is defined}\},$$

$$\bar{\Delta} = \{(f, a, f - 1_q) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid f \geqslant 1_q \text{ and } \delta(q, a) \in F\} \text{ and}$$

$$\tilde{\bar{\Delta}} = \{(f, a, f) \in \mathbb{N}_0^Q \times \Sigma \times \mathbb{N}_0^Q \mid \delta(q_0, a) \in F\}.$$

Let $A \subset (\Delta^{\sqcup\!\sqcup})^*$ be the set of all paths in $\mathbb{A}^{\sqcup\!\sqcup}$ starting with the initial state 0 and including the empty path ε. For $w \in A$, $Z(w)$ denotes the final state of the path and $Z(\varepsilon) := 0$. Formally the prefix closed language A and the function $Z : A \to \mathbb{N}_0^Q$ is defined inductively by $\varepsilon \in A$, $Z(\varepsilon) := 0$, and if $w \in A$ with $Z(w) = f$ and $(f, a, g) \in \Delta^{\sqcup\!\sqcup}$ then $w(f, a, g) \in A$ and $Z(w(f, a, g)) := g$. Let $\alpha' : (\Delta^{\sqcup\!\sqcup})^* \to \Sigma^*$ be the homomorphism with $\alpha'((f, a, g)) = a$ for $(f, a, g) \in \Delta^{\sqcup\!\sqcup}$, and let $\alpha := \alpha'_{|A}$. Hence $w \in A$ is an accepting path of a word $u \in \Sigma^*$ iff $Z(w) = 0$ and $\alpha(w) = u$.

Definition 12. $\mathbb{A}^{\sqcup\!\sqcup}$ *is called* deterministic on $w \in (\mathrm{pre}(P))^{\sqcup\!\sqcup}$, *iff* $\#(\alpha^{-1}(x)) = 1$ *for each* $x \in \mathit{pre}(w)$. *In that case, we consider* $\alpha^{-1}(x)$ *as an element of* A *instead of a subset of* A. *(*$\#(M)$ *denotes the cardinality of a set* M*)*

In [Ochsenschläger and Rieke 2010] the following theorem is proven, which will be used in Sect. 5

Theorem 4. *Let* $\mathbb{A}^{\sqcup\!\sqcup}$ *be deterministic on* $w \in (\mathit{pre}(P))^{\sqcup\!\sqcup}$, S *a countable index set and* $w'' \in SR^S_{\mathit{pre}(P)}(w)$, *then*

$$Z[\alpha^{-1}(w)](q) = \#(\{s \in S \mid \delta(q_0, \tau_s^S(w'')) = q \text{ and } \tau_s^S(w'') \notin P \cup \{\varepsilon\}\})$$

for each $q \in Q$.

Definition 13. *A prefix-closed language $L \subset \Sigma^*$ is* based deterministically on a phase $P \subset \Sigma^+$ *w.r.t.* $\mathbb{P}$*, if L is based on P and the deterministic automaton* $\mathbb{P}$ *accepts P, so that $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic on each $w \in L \subset (\mathrm{pre}(P))^{\sqcup\!\sqcup}$.*

If L is accepted by a deterministic automaton $\mathbb{L}$, then L is based deterministically on P w.r.t. $\mathbb{P}$, iff L is based on P and the product automaton of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic.

So Fig. 10 shows that L is based deterministically on P w.r.t. $\mathbb{P}$.

5. SUFFICIENT CONDITIONS FOR UNIFORMLY PARAMETERISED BEHAVIOUR PROPERTIES

We now apply theorem 3 to prove approximate satisfaction of uniformly parameterised behaviour properties. By remark 2 it remains to show simplicity of $\Pi^{IK}_{I'K'}$ on $\mathcal{L}_{IK}$. Therefore we use theorem 1, which demands for that purpose the following assumptions to be fulfilled:

Assumption 1. *There exists $\mathcal{P}_{IK} \subset \Sigma^*_{IK}$ such that*

$$\Pi^{IK}_{I'K'}(x^{-1}(\mathcal{L}_{IK})) = (\Pi^{IK}_{I'K'}(x))^{-1}(\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}))$$

for each $x \in \mathcal{P}_{IK}$.

and

Assumption 2.

$$\mathcal{L}_{IK} \subset pre(\mathcal{P}_{IK} \cap \mathcal{L}_{IK}).$$

The following definition is the key to assumpion 1.

Definition 14 (set of closed behaviours)**.** *Let $B, M \subset \Sigma^*$. M is a* set of closed behaviours *of B, iff $x^{-1}(B) = B$ for each $x \in B \cap M$.*

In Fig. 10 the initial state $(1, \emptyset)$ is the only final state of that strongly connected product automaton, so $P^{\sqcup\!\sqcup}$ is a set of closed behaviours of L.

The following theorem gives a set of closed behaviours of $\mathcal{L}_{IK}$.

Theorem 5.
Let $P^{\sqcup\!\sqcup}$ be a set of closed behaviours of L and let $\pi_\Phi(P^{\sqcup\!\sqcup})$ resp. $\pi_\Gamma(P^{\sqcup\!\sqcup})$ be a set of closed behaviours of SF resp. SG, then $\bigcap_{(i,k)\in I\times K} (\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup})$ is a set of closed behaviours of $\mathcal{L}_{IK}$.

To prove theorem 5 the following properties of left quotients are needed:

Lemma 4. *Let $h : \Sigma^* \to \Sigma'^*, A' \subset \Sigma'^*, A_i \subset \Sigma^*$ for $i \in I$ and $x \in \Sigma^*$ then*

$$x^{-1}(h^{-1}(A')) = h^{-1}((h(x))^{-1}(A')) \text{ and} \tag{22a}$$

$$x^{-1}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(x^{-1}(A_i)). \tag{22b}$$

Proof.

$$\text{(22a)} \qquad y \in x^{-1}(h^{-1}(A')) \Leftrightarrow xy \in h^{-1}(A')$$
$$\Leftrightarrow h(x)h(y) \in A'$$
$$\Leftrightarrow h(y) \in (h(x))^{-1}(A')$$
$$\Leftrightarrow y \in h^{-1}((h(x))^{-1}(A'))$$

$$\text{(22b)} \qquad y \in x^{-1}(\bigcap_{i \in I} A_i) \Leftrightarrow xy \in (\bigcap_{i \in I} A_i)$$
$$\Leftrightarrow xy \in A_i \text{ for each } i \in I$$
$$\Leftrightarrow y \in x^{-1}(A_i) \text{ for each } i \in I$$
$$\Leftrightarrow y \in \bigcap_{i \in I}(x^{-1}(A_i))$$

□

Proof. Proof for theorem 5:
Let $x \in \mathcal{L}_{IK} \cap \bigcap_{(i,k) \in I \times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})$. Then by lemma 4

$$x^{-1}(\mathcal{L}_{IK}) = x^{-1}[\bigcap_{(i,k) \in I \times K} (\pi_{ik}^{IK})^{-1}(L) \cap \bigcap_{i \in I} (\varphi_i^{IK})^{-1}(SF) \cap \bigcap_{k \in K} (\gamma_k^{IK})^{-1}(SG)]$$
$$= \bigcap_{(i,k) \in I \times K} (\pi_{ik}^{IK})^{-1}[(\pi_{ik}^{IK}(x))^{-1}(L)] \cap \bigcap_{i \in I} (\varphi_i^{IK})^{-1}[(\varphi_i^{IK}(x))^{-1}(SF)]$$
$$\cap \bigcap_{k \in K} (\gamma_k^{IK})^{-1}[(\gamma_k^{IK}(x))^{-1}(SG)].$$

So $x^{-1}(\mathcal{L}_{IK}) = \mathcal{L}_{IK}$ if for each $(i,k) \in I \times K$

$$(\pi_{ik}^{IK}(x))^{-1}(L) = L, \tag{23}$$

$$(\varphi_i^{IK}(x))^{-1}(SF) = SF, \text{ and,} \tag{24}$$

$$(\gamma_k^{IK}(x))^{-1}(SG) = SG. \tag{25}$$

By assumption

$$\pi_{ik}^{IK}(x) \in L \cap P^{\sqcup\sqcup} \text{ for } (i,k) \in I \times K \tag{26}$$

and therefore $(\pi_{ik}^{IK}(x))^{-1}(L) = L$ (23) because $P^{\sqcup\sqcup}$ is a set of closed behaviours of L.
Also by assumption

$$\varphi_r^{IK}(x) \in SF \cap \varphi_r^{IK}[\bigcap_{(i,k) \in I \times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \tag{27}$$

for $r \in I$, and

$$\gamma_s^{IK}(x) \in SG \cap \gamma_s^{IK}[\bigcap_{(i,k) \in I \times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \tag{28}$$

for $s \in K$.

To derive equations (24) and (25) from (27) and (28) it is sufficient to prove

$$\varphi_r^{IK}[\bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \subset \pi_\Phi(P^{\sqcup\sqcup}) \tag{29}$$

for $r \in I$, and

$$\gamma_s^{IK}[\bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \subset \pi_\Gamma(P^{\sqcup\sqcup}) \tag{30}$$

for $s \in K$, because $\pi_\Phi(P^{\sqcup\sqcup})$ resp. $\pi_\Gamma(P^{\sqcup\sqcup})$ is a set of closed behaviours of SF resp. SG.

The proof of (30) is analogue to (29), so it is sufficient to prove (29).

By definition $\varphi_r^{IK} = \pi_\Phi \circ \Theta^{\{r\}\times K} \circ \Pi_{\{r\}K}^{IK}$ for $r \in I$ if Σ_{rk} is identified with $\Sigma_{(r,k)}$. Therefore

$$\varphi_r^{IK}[\bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] = \pi_\Phi[\Theta^{\{r\}\times K}[\Pi_{\{r\}K}^{IK}(Y)]] \tag{31}$$

with

$$Y = \bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup}) \subset \bigcap_{k\in K}(\pi_{rk}^{IK})^{-1}(P^{\sqcup\sqcup}).$$

By definition $\pi_{rk}^{IK} = \tau_{(r,k)}^{\{r\}\times K} \circ \Pi_{\{r\}K}^{IK}$ for $r \in I$ and $k \in K$, if Σ_{rk} is identified with $\Sigma_{(r,k)}$. Therefore

$$\bigcap_{k\in K}(\pi_{rk}^{IK})^{-1}(P^{\sqcup\sqcup}) = (\Pi_{\{r\}K}^{IK})^{-1}[\bigcap_{k\in K}(\tau_{(r,k)}^{\{r\}\times K})^{-1}(P^{\sqcup\sqcup})],$$

which implies $\Pi_{\{r\}K}^{IK}(Y) \subset \bigcap_{k\in K}(\tau_{(r,k)}^{\{r\}\times K})^{-1}(P^{\sqcup\sqcup})$.

Now by (31) we have

$$\varphi_r^{IK}[\bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \subset \pi_\Phi[\Theta^{\{r\}\times K}[\bigcap_{k\in K}(\tau_{(r,k)}^{\{r\}\times K})^{-1}(P^{\sqcup\sqcup})]]. \tag{32}$$

By definition of $P^{\sqcup\sqcup}$, lemma 1 and lemma 2 we have

$$\begin{aligned}\Theta^{\{r\}\times K}[\bigcap_{k\in K}(\tau_{(r,k)}^{\{r\}\times K})^{-1}(P^{\sqcup\sqcup})] &= \Theta^{\{r\}\times K}[\bigcap_{k\in K}(\tau_{(r,k)}^{\{r\}\times K})^{-1}(\Theta^{\mathbb{N}}[\bigcap_{t\in\mathbb{N}}(\tau_t^{\mathbb{N}})^{-1}(P\cup\{\varepsilon\})])] \\ &= \Theta^{\{r\}\times K\times\mathbb{N}}[\bigcap_{(k,t)\in K\times\mathbb{N}}(\tau_{(k,t)}^{\{r\}\times K\times\mathbb{N}})^{-1}(P\cup\{\varepsilon\})] \\ &= P^{\sqcup\sqcup}.\end{aligned}$$

Now (32) implies

$$\varphi_r^{IK}[\bigcap_{(i,k)\in I\times K}(\pi_{ik}^{IK})^{-1}(P^{\sqcup\sqcup})] \subset \pi_\Phi(P^{\sqcup\sqcup}),$$

which proves (29) and therefore completes the proof of theorem 5. □

To check for example if $\pi_\Phi(P^{\sqcup\sqcup})$ is a set of closed behaviours of SF (see Fig. 11(b)) the following result is helpful:

Theorem 6 (homomorphism theorem for $P^{\sqcup}$)**.**
Let $\mu : \Sigma^* \to \Sigma'^*$ be an alphabetic homomorphism, then holds $\mu(P^{\sqcup}) = (\mu(P))^{\sqcup}$.

Proof. Let $\mu_{\mathbb{N}} : \Sigma_{\mathbb{N}}^* \to \Sigma_{\mathbb{N}}'^*$ the homomorphism with

$$\mu_{\mathbb{N}}(a_t) := (\mu(a))_t$$

for $a_t \in \Sigma_t$ and $t \in \mathbb{N}$, where $(\varepsilon)_t = \varepsilon$.
Let $\tau_t'^{\mathbb{N}} : \Sigma_{\mathbb{N}}'^* \to \Sigma'^*$ and $\Theta'^{\mathbb{N}} : \Sigma_{\mathbb{N}}'^* \to \Sigma'^*$ be defined like $\tau_t^{\mathbb{N}}$ and $\Theta^{\mathbb{N}}$.
For these homomorphisms holds $\mu \circ \Theta^{\mathbb{N}} = \Theta'^{\mathbb{N}} \circ \mu_{\mathbb{N}}$, and therewith

$$\mu(P^{\sqcup}) = \Theta'^{\mathbb{N}}[\mu_{\mathbb{N}}(\bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\}))]. \tag{33}$$

From this it follows that $\mu(P^{\sqcup}) = (\mu(P))^{\sqcup}$ if the following equation holds:

$$\mu_{\mathbb{N}}(\bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})) = \bigcap_{t\in\mathbb{N}} (\tau_t'^{\mathbb{N}})^{-1}(\mu(P) \cup \{\varepsilon\}) \tag{34}$$

Proof. Proof of equation (34):
For each $t \in \mathbb{N}$ holds $\tau_t'^{\mathbb{N}} \circ \mu_{\mathbb{N}} = \mu \circ \tau_t^{\mathbb{N}}$.
For $x \in \bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})$ and $t \in \mathbb{N}$ from this it follows:

$$\tau_t'^{\mathbb{N}}(\mu_{\mathbb{N}}(x)) = \mu(\tau_t^{\mathbb{N}}(x)) \in \mu(P \cup \{\varepsilon\}) = \mu(P) \cup \{\varepsilon\},$$

and so

$$\mu_{\mathbb{N}}(\bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})) \subset \bigcap_{t\in\mathbb{N}} (\tau_t'^{\mathbb{N}})^{-1}(\mu(P) \cup \{\varepsilon\}).$$

For the proof of the other inclusion of equation(34) we now prove the following proposition:
For each $y \in \Sigma_{\mathbb{N}}'^*$ and $(u_t)_{t\in\mathbb{N}}$ with $\tau_t'^{\mathbb{N}}(y) = \mu(u_t)$, $u_t \in \Sigma^+$ for $t \in T(y)$ and $u_t = \varepsilon$ for $t \in \mathbb{N} \setminus T(y)$ exists an $x \in \Sigma_{\mathbb{N}}^*$ with $y = \mu_{\mathbb{N}}(x)$ and $\tau_t^{\mathbb{N}}(x) = u_t$ for each $t \in \mathbb{N}$.
Thereby is $T(y) := \{t \in \mathbb{N} \mid \tau_t'^{\mathbb{N}}(y) \neq \varepsilon\}$, hence $T(y)$ is a finite set.

Proof by induction.
Induction base.
For $y = \varepsilon$ holds $T(y) = \emptyset$, and $x = \varepsilon$ satisfies the proposition.
Induction step.
Let $y = y'a_s' \in \Sigma_{\mathbb{N}}'^*$ with $a_s' \in \Sigma_s'$ and $\tau_t'^{\mathbb{N}}(y) = \mu(u_t)$ with $u_t \in \Sigma^+$ for $t \in T(y)$ as well as $u_t = \varepsilon$ for $t \in \mathbb{N} \setminus T(y)$.
Then holds $s \in T(y)$, because $\tau_s'^{\mathbb{N}}(y) = \tau_s'^{\mathbb{N}}(y')a_s' \neq \varepsilon$.
Let now $u_s = u_s'v_s'$ with $v_s' \in \Sigma^+$, $a_s' = \tau_s'^{\mathbb{N}}(a_s') = \mu(v_s') \neq \varepsilon$ and $u_s' = \varepsilon$ when $\tau_s'^{\mathbb{N}}(y') = \varepsilon$.
For $t \in \mathbb{N} \setminus \{s\}$ let $u_t' := u_t$.
$y' \in \Sigma_{\mathbb{N}}'^*$ and $(u_t')_{t\in\mathbb{N}}$ now satisfy the induction hypothesis. Therefore exists $x' \in \Sigma_{\mathbb{N}}^*$ with $y' = \mu_{\mathbb{N}}(x')$ and $\tau_t^{\mathbb{N}}(x') = u_t'$ for each $t \in \mathbb{N}$.
Because of the injectivity of $\tau_s^{\mathbb{N}}$ on Σ_s^* exists now exactly one $\tilde{v}_s \in \Sigma_s^+$ with $\tau_s^{\mathbb{N}}(\tilde{v}_s) = v_s'$.

According to the definition of $\mu_{\mathbb{N}}$ now for $\tilde{v}_s$ holds:
$\mu_{\mathbb{N}}(\tilde{v}_s) = a'_s$, hence $\mu_{\mathbb{N}}(x'\tilde{v}_s) = \mu_{\mathbb{N}}(x')\mu_{\mathbb{N}}(\tilde{v}_s) = y'a'_s = y$.
Because $\tau_t^{\mathbb{N}}(x'\tilde{v}_s) = \tau_t^{\mathbb{N}}(x') = u'_t = u_t$ for $t \in \mathbb{N}\setminus\{s\}$ and $\tau_s^{\mathbb{N}}(x'\tilde{v}_s) = \tau_s^{\mathbb{N}}(x')\tau_s^{\mathbb{N}}(\tilde{v}_s) = u'_s v'_s = u_s$ is then $x := x'\tilde{v}_s$ a proper $x \in \Sigma_{\mathbb{N}}^*$ for $y = y'a'_s \in \Sigma_{\mathbb{N}}'^*$ for the induction step. Therewith the proof of the proposition is completed. □

From the above proposition follows the inclusion

$$\bigcap_{t\in\mathbb{N}} (\tau_t'^{\mathbb{N}})^{-1}(\mu(P) \cup \{\varepsilon\}) \subset \mu_{\mathbb{N}}(\bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(P \cup \{\varepsilon\})),$$

which completes the proof of equation (34). □

This in turn completes the proof of the homomorphism theorem 6 for $P^{\sqcup}$. □

The proofs of (33) and (34) do not depend on the special index set $\mathbb{N}$. (33) and (34) hold for arbitrary index sets S (note that we assume index sets beeing not empty), which imply a corollary for structural representations.
For $x \in \Sigma^*$ and $u \in \mathrm{SR}^S_{P\cup\{\varepsilon\}}(x)$ holds $x = \Theta^S(u)$ and $u \in \bigcap_{t\in S} (\tau_t^S)^{-1}(P\cup\{\varepsilon\})$. Now (33) and (34) imply $\Theta'^S(\mu_S(u)) = \mu(\Theta^S(u)) = \mu(x)$ and $\mu_S(u) \in \mu_S(\bigcap_{t\in S} (\tau_t^S)^{-1}(P\cup \{\varepsilon\})) = \bigcap_{t\in S} (\tau_t'^S)^{-1}(\mu(P) \cup \{\varepsilon\})$. So we get $\mu_S(u) \in \mathrm{SR}^S_{\mu(P)\cup\{\varepsilon\}}(\mu(x))$ and

Corollary 1. $\mu_S(\mathrm{SR}^S_{P\cup\{\varepsilon\}}(x)) \subset \mathrm{SR}^S_{\mu(P)\cup\{\varepsilon\}}(\mu(x))$.

The following theorem states that $\mathcal{P}_{IK} := \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup})$ fulfills assumption 1. This $\mathcal{P}_{IK}$ consists of all elements of Σ_{IK}^* “in which all phases are completed”.

Theorem 7.
Let $\mathcal{L}_{IK}$ be self-similar and let the assumptions of theorem 5 be fulfilled, then

$$\Pi^{IK}_{I'K'}(x^{-1}(\mathcal{L}_{IK})) = (\Pi^{IK}_{I'K'}(x))^{-1}(\Pi^{IK}_{I'K'}(\mathcal{L}_{IK}))$$

for each $x \in \mathcal{L}_{IK} \cap \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup})$ and $I' \times K' \subset I \times K$.

For its proof we need the following

Lemma 5. *For $I' \subset I$, $K' \subset K$, and $L \subset \Sigma^*$ with $\varepsilon \in L$, the following relationships hold:*

$$\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] = (\pi_{ik}^{I'K'})^{-1}(L) \text{ for } (i,k) \in I' \times K', \tag{35a}$$

$$\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] = \Sigma^*_{I'K'} \text{ for } (i,k) \in (I \times K) \setminus (I' \times K'). \tag{35b}$$

Proof.

(35a) $x \in \Sigma^*_{I'K'}$ and $\pi_{ik}^{I'K'}(x) \in L$, for $x \in (\pi_{ik}^{I'K'})^{-1}(L)$. From this it follows that $x \in \Sigma^*_{IK}$, $\pi_{ik}^{IK}(x) = \pi_{ik}^{I'K'}(x) \in L$ and $x = \Pi^{IK}_{I'K'}(x)$, which implies $x \in \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$. Hence $(\pi_{ik}^{I'K'})^{-1}(L) \subset \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$. For $x \in \Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)]$ exists $y \in \Sigma^*_{IK}$ such that $\pi_{ik}^{IK}(y) \in L$ and $x = \Pi^{IK}_{I'K'}(y)$. Since $(i,k) \in I' \times K'$ it follows that $\pi_{ik}^{IK}(y) = \pi_{ik}^{I'K'}(\Pi^{IK}_{I'K'}(y)) = \pi_{ik}^{I'K'}(x) \in L$ which proves the inclusion $\Pi^{IK}_{I'K'}[(\pi_{ik}^{IK})^{-1}(L)] \subset (\pi_{ik}^{I'K'})^{-1}(L)$.

(35b) For $x \in \Sigma^*_{I'K'}$ and $(i,k) \in (I \times K) \setminus (I' \times K')$ holds $x \in \Sigma^*_{IK}$, $\pi^{IK}_{ik}(x) = \varepsilon \in L$ and $x = \Pi^{IK}_{I'K'}(x)$, and so $x \in \Pi^{IK}_{I'K'}[(\pi^{IK}_{ik})^{-1}(L)]$. Hence $\Sigma^*_{I'K'} \subset \Pi^{IK}_{I'K'}[(\pi^{IK}_{ik})^{-1}(L)]$. The reverse inclusion holds because of $\Pi^{IK}_{I'K'} : \Sigma^*_{IK} \to \Sigma^*_{I'K'}$.

□

Proof. By self-similarity of $\mathcal{L}_{IK}$ and theorem 5 holds

$$\Pi^{IK}_{I'K'}(x^{-1}(\mathcal{L}_{IK})) = \mathcal{L}_{I'K'}. \tag{36}$$

By (35a) and (35b) we have

$$\begin{aligned}
\Pi^{IK}_{I'K'}(x) &\in \Pi^{IK}_{I'K'}[\mathcal{L}_{IK} \cap \bigcap_{(i,k)\in I\times K} (\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup})] \\
&\subset \Pi^{IK}_{I'K'}(\mathcal{L}_{IK}) \cap \bigcap_{(i,k)\in I'\times K'} \Pi^{IK}_{I'K'}((\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup})) \\
&\quad \cap \bigcap_{(i,k)\in (I\times K)\setminus(I'\times K')} \Pi^{IK}_{I'K'}((\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup})) \\
&= \mathcal{L}_{I'K'} \cap \bigcap_{(i,k)\in I'\times K'} (\pi^{I'K'}_{ik})^{-1}(P^{\sqcup\!\sqcup}) \cap \Sigma^*_{I'K'} \\
&= \mathcal{L}_{I'K'} \cap \bigcap_{(i,k)\in I'\times K'} (\pi^{I'K'}_{ik})^{-1}(P^{\sqcup\!\sqcup}).
\end{aligned}$$

Now self-similarity of $\mathcal{L}_{IK}$ and theorem 5 imply

$$(\Pi^{IK}_{I'K'}(x))^{-1}(\Pi^{IK}_{I'K'}(\mathcal{L}_{IK})) = (\Pi^{IK}_{I'K'}(x))^{-1}(\mathcal{L}_{I'K'}) = \mathcal{L}_{I'K'}.$$

Together with (36) this proves theorem 7. □

We now formulate conditions to fulfill assumption 2.

Condition I. *Following the ideas of [Ochsenschläger and Rieke 2010] we assume that $L \subset \Sigma^*$ is deterministically based on a phase $P \subset \Sigma^+$ w.r.t. a deterministic automaton $\mathbb{P}$ accepting P such that $P^{\sqcup\!\sqcup}$ is a set of closed behaviours of L.*

By condition I for $w \in \mathcal{L}_{IK}$ and $w \notin \bigcap_{(i,k)\in I\times K} (\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup})$ there exists $(r,s) \in I \times K$ with $\pi^{IK}_{rs}(w) \in L \cap \text{pre}(P^{\sqcup\!\sqcup})$ and $\pi^{IK}_{rs}(w) \notin P^{\sqcup\!\sqcup}$.

If Q_P is the set of states of $\mathbb{P}$ then by definition of $\mathbb{P}^{\sqcup\!\sqcup}$ for $y \in \text{pre}(P^{\sqcup\!\sqcup}) \cap L$, $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic on y and therefore $\alpha_P^{-1}(y)$ consists of exactly one element.

$$\mathcal{D}_{\mathbb{P}}(y) := \sum_{q\in Q_P} [Z_P(\alpha_P^{-1}(y))](q) \in \mathbb{N}_0 \tag{37}$$

is the number of "open phases in y" where Z_P and α_P denote the Z- and α-functions of $\mathbb{P}^{\sqcup\!\sqcup}$. Therefore it describes the "defect" of an $y \in \text{pre}(P^{\sqcup\!\sqcup}) \cap L$ relative to $P^{\sqcup\!\sqcup} \cap L$. The index $\mathbb{P}$ in (37) denotes that Q_P and the functions Z_P and α_P depend on the automaton $\mathbb{P}$ resp. $\mathbb{P}^{\sqcup\!\sqcup}$.

In particular $y \in P^{\sqcup\!\sqcup} \cap L$ iff $y \in \text{pre}(P^{\sqcup\!\sqcup}) \cap L$ and $\mathcal{D}_{\mathbb{P}}(y) = 0$.

So we have

Theorem 8.
Let condition I be fulfilled and for each $w \in \mathcal{L}_{IK} \setminus \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup\!\sqcup})$ *exists* $(r,s) \in I \times K$ *and* $v \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^{+}$ *with* $\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(wv)) < \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w))$ *then for each* $w \in \mathcal{L}_{IK}$ *there exists* $u \in w^{-1}(\mathcal{L}_{IK})$ *with* $wu \in \mathcal{L}_{IK} \cap \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(P^{\sqcup\!\sqcup})$.

Proof. Iterated application of the hypothesis of theorem 8 eventually leads to $wv_1 \dots v_n \in \mathcal{L}_{IK}$ with $\mathcal{D}((\pi_{ik}^{IK}(wv_1 \dots v_n)) = 0$ for each $(i,k) \in I \times K$. Therewith follows the conclusion of the theorem for $u = v_1 \dots v_n$. □

Theorem 8 is the base to construct step by step for each $w \in \mathcal{L}_{IK}$ a $v \in \mathcal{L}_{IK} \cap \mathcal{P}_{IK}$ with $w \in \text{pre}(v)$.

We now consider example 2. Let $P \subset \Sigma^{+}$ be defined by the automaton $\mathbb{P}$ in Fig. 6. As shown in Sect. 4 (Fig. 10) L is based deterministically on phase P w.r.t. $\mathbb{P}$ and as mentioned after definition 14 $P^{\sqcup\!\sqcup}$ is a set of closed behaviours of L. So condition I is fulfilled in example 2.

The automaton $\mathbb{PF}$ in Fig. 11(a) is the minimal automaton of $\pi_{\Phi}(P) \subset \Phi^{+}$.

By theorem 6 $SF \cap \pi_{\Phi}(P^{\sqcup\!\sqcup}) = SF \cap (\pi_{\Phi}(P))^{\sqcup\!\sqcup}$. So $SF \cap \pi_{\Phi}(P^{\sqcup\!\sqcup})$ is accepted by the product automaton of $\mathbb{SF}$ and $\mathbb{PF}^{\sqcup\!\sqcup}$ which is depicted in Fig. 11(b).

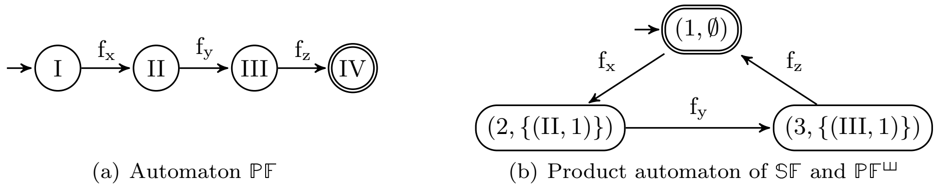


Fig. 11. Automaton $\mathbb{PF}$ and product automaton of $\mathbb{SF}$ and $\mathbb{PF}^{\sqcup\!\sqcup}$

By the same argument as for the product automaton of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$ SF is based deterministically on $\pi_{\Phi}(P)$ w.r.t. $\mathbb{PF}$, and $\pi_{\Phi}(P^{\sqcup\!\sqcup})$ is a set of closed behaviours of SF.

The automaton $\mathbb{PG}$ in Fig. 12(a) is the minimal automaton of $\pi_{\Gamma}(P) \subset \Gamma^{+}$.

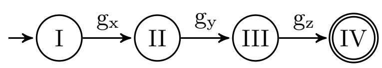
By theorem 6 $SG \cap \pi_{\Gamma}(P^{\sqcup\!\sqcup}) = SG \cap (\pi_{\Gamma}(P))^{\sqcup\!\sqcup}$. So $SG \cap \pi_{\Gamma}(P^{\sqcup\!\sqcup})$ is accepted by the product automaton of $\mathbb{SG}$ and $\mathbb{PG}^{\sqcup\!\sqcup}$ which is depicted in Fig. 12(b).

By the same argument as for the product automaton of $\mathbb{L}$ and $\mathbb{P}^{\sqcup\!\sqcup}$ SG is based deterministically on $\pi_{\Gamma}(P)$ w.r.t. $\mathbb{PG}$, and $\pi_{\Gamma}(P^{\sqcup\!\sqcup})$ is a set of closed behaviours of SG.

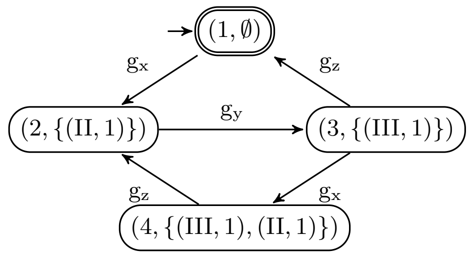
So especially all assumptions of theorem 5 and 8 are fulfilled for this example, because in the appendix self-similarity of $\mathcal{L}_{IK}$ has been proven.

The automata of Fig. 6, Fig. 11(b) and Fig. 12(b) show that

- each phase is initiated by an F-action,
- each F-partner is “involved” in at most one phase, and
- each G-partner is “involved” in at most two phases.



(a) Automaton $\mathbb{PG}$



(b) Product automaton of $\mathbb{SG}$ and $\mathbb{PG}^{\sqcup\!\sqcup}$

Fig. 12. Automaton $\mathbb{PG}$ and product automaton of $\mathbb{SG}$ and $\mathbb{PG}^{\sqcup\!\sqcup}$

To construct the completions of phases v of theorem 8 one may imagine that the following strategy could work.

Completion strategy:

(1) For each G-partner "involved" in two phases "complete" one of this phases.

(2) For each G-partner "involved" in one phases "complete" this phase.

(3) "Complete" the phases, where only an F-partner is "involved" in.

To formalise such a strategy more generally and to make corresponding completions of phases possible, some preparations and additional conditions are needed. These conditions, including condition I, we call *success conditions* for the completion strategy.

Condition II. *To formalise the "number of phases a partner is involved in" we now assume that*

(i) $SF \subset \Phi^*$ *is deterministically based on* $\pi_\Phi(P) \subset \Phi^+$ *w.r.t. the minimal automaton* $\mathbb{PF}$ *of* $\pi_\Phi(P)$ *,*

(ii) $\pi_\Phi(P^{\sqcup\!\sqcup})$ *is a set of closed behaviours of* SF *,*

(iii) $SG \subset \Gamma^*$ *is deterministically based on* $\pi_\Gamma(P) \subset \Gamma^+$ *w.r.t. the minimal automaton* $\mathbb{PG}$ *of* $\pi_\Gamma(P)$ *and*

(iv) $\pi_\Gamma(P^{\sqcup\!\sqcup})$ *is a set of closed behaviours of* SG*.*

Fig. 11(b) and Fig. 12(b) show that condition II is fulfilled in example 2.

For each $w \in \mathcal{L}_{IK}$, $i \in I$ and $k \in K$ holds $\varphi_i^{IK}(w) \in SF$ and $\gamma_k^{IK}(w) \in SG$.

By condition II $SF = SF \cap \mathrm{pre}((\pi_\Phi(P))^{\sqcup\!\sqcup})$ resp. $SG = SG \cap \mathrm{pre}((\pi_\Gamma(P))^{\sqcup\!\sqcup})$ and $\mathbb{PF}^{\sqcup\!\sqcup}$ resp. $\mathbb{PG}^{\sqcup\!\sqcup}$ is deterministic on $\varphi_i^{IK}(w)$ resp. $\gamma_k^{IK}(w)$.

Now $\mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w))$ resp. $\mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w))$ formally defines the "number of phases partner i resp. k is involved in". $\mathcal{D}_{\mathbb{PF}}$ and $\mathcal{D}_{\mathbb{PG}}$ are defined analogously to $\mathcal{D}_{\mathbb{P}}$:

For $y \in SF \cap \mathrm{pre}((\pi_\Phi(P))^{\sqcup\!\sqcup})$

$$\mathcal{D}_{\mathbb{PF}}(y) := \sum_{q \in Q_{PF}} [Z_{PF}(\alpha_{PF}^{-1}(y))](q)$$

and for $y \in SG \cap \mathrm{pre}((\pi_\Gamma(P))^{\sqcup\!\sqcup})$

$$\mathcal{D}_{\mathbb{PG}}(y) := \sum_{q \in Q_{PG}} [Z_{PG}(\alpha_{PG}^{-1}(y))](q),$$

where Q_{PF} resp. Q_{PG} is the state set of $\mathbb{PF}$ resp. $\mathbb{PG}$ and Z_{PF} resp. Z_{PG} and α_{PF} resp. α_{PG} denote the Z- and α-function of $\mathbb{PF}^{\sqcup}$ resp. $\mathbb{PG}^{\sqcup}$.

We now want to derive relations between $\mathcal{D}_{\mathbb{P}}(\pi_{ik}^{IK}(w))$ and $\mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w))$ resp. $\mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w))$ for $w \in \mathcal{L}_{IK}$, $i \in I$ and $k \in K$.

As in the proof of theorem 5 the following holds:

$$\mathcal{L}_{IK} \subset \bigcap_{(i,k)\in I\times K} (\pi_{ik}^{IK})^{-1}(L) \subset \bigcap_{k\in K} (\pi_{rk}^{IK})^{-1}(L)$$

for each $r \in I$.
On account of $\pi_{rk}^{IK} = \tau_{(r,k)}^{\{r\}\times K} \circ \Pi_{\{r\}K}^{IK}$ we get

$$\mathcal{L}_{IK} \subset (\Pi_{\{r\}K}^{IK})^{-1}[\bigcap_{k\in K} (\tau_{(r,k)}^{\{r\}\times K})^{-1}(L)],$$

which implies

$$\Pi_{\{r\}K}^{IK}(w) \in \bigcap_{k\in K} (\tau_{(r,k)}^{\{r\}\times K})^{-1}(L) \tag{38}$$

for each $w \in \mathcal{L}_{IK}$ and $r \in I$.

By $\varphi_r^{IK} = \pi_\Phi \circ \Theta^{\{r\}\times K} \circ \Pi_{\{r\}K}^{IK}$ (38) implies

$$\varphi_r^{IK}(w) \subset \pi_\Phi[\Theta^{\{r\}\times K}[\bigcap_{k\in K} (\tau_{(r,k)}^{\{r\}\times K})^{-1}(L)]] \tag{39}$$

for each $w \in \mathcal{L}_{IK}$.

Condition I implies

$$L = \mathrm{pre}(L \cap P^{\sqcup}) \subset \mathrm{pre}(P^{\sqcup}) = (\mathrm{pre}(P))^{\sqcup}.$$

Now by lemma 1 and 2

$$\begin{aligned}\Pi_{\{r\}K}^{IK}(w) &\in \bigcap_{k\in K} (\tau_{(r,k)}^{\{r\}\times K})^{-1}(L)\\ &\subset \bigcap_{k\in K} (\tau_{(r,k)}^{\{r\}\times K})^{-1}[\Theta^{\mathbb{N}}(\bigcap_{t\in\mathbb{N}} (\tau_t^{\mathbb{N}})^{-1}(\mathrm{pre}(P)))]\\ &= \Theta_{\{r\}\times K}^{\{r\}\times K\times\mathbb{N}}[\bigcap_{(k,t)\in K\times\mathbb{N}} (\tau_{(r,k,t)}^{\{r\}\times K\times\mathbb{N}})^{-1}(\mathrm{pre}(P))]\end{aligned} \tag{40}$$

which implies

$$w' := \Theta^{\{r\}\times K}(\Pi_{\{r\}K}^{IK}(w)) \in (\mathrm{pre}(P))^{\sqcup} \tag{41}$$

on account of $\Theta^{\{r\}\times K} \circ \Theta_{\{r\}\times K}^{\{r\}\times K\times\mathbb{N}} = \Theta^{\{r\}\times K\times\mathbb{N}}$.

So by (38)-(41) for each $w \in \mathcal{L}_{IK}$ and $r \in I$ there exists $w'' \in \mathrm{SR}_{\mathrm{pre}(P)}^{\{r\}\times K\times\mathbb{N}}(w')$ with

$$\varphi_r^{IK}(w) = \pi_\Phi(w') \text{ and } \Pi_{\{r\}K}^{IK}(w) = \Theta_{\{r\}\times K}^{\{r\}\times K\times\mathbb{N}}(w''). \tag{42}$$

Now by corollary 1

$$\pi_{\Phi_{\{r\}\times K\times\mathbb{N}}}(w'') \in \mathrm{SR}_{\pi_\Phi(\mathrm{pre}(P))}^{\{r\}\times K\times\mathbb{N}}(\varphi_r^{IK}(w)) \tag{43}$$

where

$$\pi_{\Phi_{\{r\}\times K\times\mathbb{N}}} : \Sigma^*_{\{r\}\times K\times\mathbb{N}} \to \Phi^*_{\{r\}\times K\times\mathbb{N}} \text{ with } \pi_{\Phi_{\{r\}\times K\times\mathbb{N}}}(a_s) := (\pi_\Phi(a))_s$$

for $a_s \in \Sigma_s$, $s \in \{r\} \times K \times \mathbb{N}$ and $(\varepsilon)_s := \varepsilon$.

By condition II $\mathbb{PF}^{\sqcup}$ is deterministic on $\varphi_r^{IK}(w)$ for $w \in \mathcal{L}_{IK}$ and $r \in I$. Therefore theorem 4 applies to $\mathbb{PF}^{\sqcup}$ with the structural representation of (43) and we get

$$\begin{aligned} Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))](q) =& \#(\{(r,k,t) \in \{r\} \times K \times \mathbb{N} \mid \\ & \delta_{PF}(q_{PF0}, \tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(\pi_{\Phi_{\{r\}\times K\times\mathbb{N}}}(w''))) = q \\ & \text{and } \tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(\pi_{\Phi_{\{r\}\times K\times\mathbb{N}}}(w'')) \notin \pi_\Phi(P) \cup \{\varepsilon\}\}) \\ & \text{for each } q \in Q_{PF}. \end{aligned} \tag{44}$$

Here Q_{PF} is the state set, q_{PF0} the initial state and δ_{PF} the state transition function of $\mathbb{PF}$. Z_{PF} and α_{PF} denote the corresponding Z- and α-functions of $\mathbb{PF}$.

By the proof of theorem 6 and corollary 1 the homomorphisms $\tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)} : \Phi^*_{\{r\}\times K\times\mathbb{N}} \to \Phi^*$ are defined by

$$\tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(x) := \tau^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(x) \tag{45}$$

for each $x \in \Phi^*_{\{r\}\times K\times\mathbb{N}} \subset \Sigma^*_{\{r\}\times K\times\mathbb{N}}$.
As mentioned in the proof of (34)

$$\tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)} \circ \pi_{\Phi_{\{r\}\times K\times\mathbb{N}}} = \pi_\Phi \circ \tau^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)},$$

which implies

$$\tau'^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(\pi_{\Phi_{\{r\}\times K\times\mathbb{N}}}(w'')) = \pi_\Phi(\tau^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(w'')) \tag{46}$$

for each $k \in K$ and $t \in \mathbb{N}$.

Now we apply lemma 3 to the structural representation w'' of (42). Let $k \in K$, $M := \text{pre}(P)$, $S := \{r\} \times K$, $T := \mathbb{N}$, $y' := w''$ and $x := \Pi^{IK}_{\{r\}K}(w)$. By (42) all assumptions of lemma 3 are fulfilled and hence

$$\begin{aligned} \Pi^{\{r\}\times K\times\mathbb{N}}_{\{r\}\times\{k\}\times\mathbb{N}}(w'') \in& \mathrm{SR}^{\{r\}\times\{k\}\times\mathbb{N}}_{\text{pre}(P)}(\Theta^{\{r\}\times\{k\}}(\Pi^{\{r\}\times K}_{\{r\}\times\{k\}}(\Pi^{IK}_{\{r\}\times K}(w)))) \\ =& \mathrm{SR}^{\{r\}\times\{k\}\times\mathbb{N}}_{\text{pre}(P)}(\pi^{IK}_{rk}(w)) \end{aligned} \tag{47}$$

by identifying $\Sigma_{\{r\}K}$ with $\Sigma_{\{r\}\times K}$ and $\Sigma_{\{r\}\times\{k\}}$ with Σ_{rk}.

(44) and (46) imply

$$\begin{aligned} Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))](q) =& \#(\{(r,k,t) \in \{r\} \times K \times \mathbb{N} \mid \\ & \delta_{PF}(q_{PF0}, \pi_\Phi(\tau^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(w''))) = q \\ & \text{and } \pi_\Phi(\tau^{\{r\}\times K\times\mathbb{N}}_{(r,k,t)}(w'')) \notin \pi_\Phi(P) \cup \{\varepsilon\}\}) \\ & \text{for each } q \in Q_{PF}. \end{aligned} \tag{48}$$

By condition I $\mathbb{P}^{\sqcup}$ is deterministic on $\pi^{IK}_{rk}(w)$ for $w \in \mathcal{L}_{IK}$ and $(r,k) \in I \times K$. Therefore theorem 4 applies to $\mathbb{P}^{\sqcup}$ with the structural representation of (47) and we get

$$\begin{aligned} Z_P[\alpha_P^{-1}(\pi_{rk}^{IK}(w))](q) =& \#(\{(r,k,t) \in \{r\} \times \{k\} \times \mathbb{N} \mid \\ & \delta_P(q_{P0'}, \tau_{(r,k,t)}^{\{r\}\times\{k\}\times\mathbb{N}}(\Pi_{\{r\}\times\{k\}\times\mathbb{N}}^{\{r\}\times K\times\mathbb{N}}(w''))) = q \\ & \text{and } \tau_{(r,k,t)}^{\{r\}\times\{k\}\times\mathbb{N}}(\Pi_{\{r\}\times\{k\}\times\mathbb{N}}^{\{r\}\times K\times\mathbb{N}}(w'')) \notin P \cup \{\varepsilon\}\}) \\ =& \#(\{(r,k,t) \in \{r\} \times \{k\} \times \mathbb{N} \mid \\ & \delta_P(q_{P0'}, \tau_{(r,k,t)}^{\{r\}\times K\times\mathbb{N}}(w'')) = q \\ & \text{and } \tau_{(r,k,t)}^{\{r\}\times K\times\mathbb{N}}(w'') \notin P \cup \{\varepsilon\}\}) \\ & \text{for each } q \in Q_P \text{ and } k \in K \end{aligned} \tag{49}$$

because of

$$\tau_{(r,k,t)}^{\{r\}\times\{k\}\times\mathbb{N}} \circ \Pi_{\{r\}\times\{k\}\times\mathbb{N}}^{\{r\}\times K\times\mathbb{N}} = \tau_{(r,k,t)}^{\{r\}\times K\times\mathbb{N}}$$

for each $(r,k,t) \in I \times K \times \mathbb{N}$.

In (49) Q_P is the state set, q_{P0} the initial state and δ_P the state transition function of $\mathbb{P}$.

By (47) $\tau_{(r,k,t)}^{\{r\}\times K\times\mathbb{N}}(w'') \in \text{pre}(P)$ for each $(r,k,t) \in I \times K \times \mathbb{N}$.

Now equations (48) and (49) imply relations between $Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))]$ and $Z_P[\alpha_P^{-1}(\pi_{rk}^{IK}(w))]$ which can be used to formulate conditions allowing completions of phases to reduce $\mathcal{D}_{\mathbb{P}}(\pi_{rk}^{IK}(w))$.

For that purpose let the relation $R_\Phi \subset Q_P \times Q_{PF}$ be defined by

$$R_\Phi := \{(\delta_P(q_{P0}, u)), \delta_{PF}(q_{PF0}, \pi_\Phi(u))) \in Q_P \times Q_{PF} |\ u \in \text{pre}(P)\}.$$

For $q \in Q_P$ and $q_F \in Q_{PF}$ we also use the notation

$$R_\Phi(q) := \{x \in Q_{PF} |\ (q,x) \in R_\Phi\} \text{ and } R_\Phi^{-1}(q_F) := \{y \in Q_P |\ (y, q_F) \in R_\Phi\}.$$

If $Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))](q_F) > 0$ for some $q_F \in Q_{PF}$ then by (48) there exists $k_q \in K$ and $t_q \in \mathbb{N}$ such that $\delta_{PF}(q_{PF0}, \pi_\Phi(\tau_{(r,k_q,t_q)}^{\{r\}\times K\times\mathbb{N}}(w''))) = q_F$ and $\pi_\Phi(\tau_{(r,k_q,t_q)}^{\{r\}\times K\times\mathbb{N}}(w'')) \notin \pi_\Phi(P) \cup \{\varepsilon\}$. This implies $\tau_{(r,k_q,t_q)}^{\{r\}\times K\times\mathbb{N}}(w'') \notin P \cup \{\varepsilon\}$ and by (49) there exist $q \in R_\Phi^{-1}(q_F)$ with

$$Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))](q) > 0. \tag{50}$$

Concerning an implication in the other direction one have to note that $u \in \text{pre}(P)$ and $u \notin P \cup \{\varepsilon\}$ does not imply $\pi_\Phi(u) \notin \pi_\Phi(P) \cup \{\varepsilon\}$. For that reason let

$$E_\Phi := \{\delta_P(q_{P0}, u) \in Q_P \mid u \in (\text{pre}(P) \setminus \{\varepsilon\}) \cap \pi_\Phi^{-1}(\{\varepsilon\})\} \text{ and}$$

$$P_\Phi := \{\delta_P(q_{P0}, u) \in Q_P \mid u \in (\text{pre}(P) \setminus P) \cap \pi_\Phi^{-1}(\pi_\Phi(P))\}.$$

If $Z_P[\alpha_P^{-1}(\pi_{rk}^{IK}(w))](p) > 0$ for some $p \in Q_P$ and $k \in K$ then by (49) there exists $t_p \in \mathbb{N}$ such that $\delta_P(q_{P0}, \tau_{(r,k,t_p)}^{\{r\}\times K\times\mathbb{N}}(w'')) = p$ and $\tau_{(r,k,t_p)}^{\{r\}\times K\times\mathbb{N}}(w'') \notin P \cup \{\varepsilon\}$.

Now if $p \in E_\Phi \cup P_\Phi$ by (48) there exists $p_F \in R_\Phi(p)$ with

$$Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))](p_F) > 0. \tag{51}$$

By the same argumentation and corresponding definitions of R_Γ, E_Γ and P_Γ one gets corresponding propositions for $\gamma_s^{IK}(w)$ with $s \in K$ and $w \in \mathcal{L}_{IK}$:
If $Z_{PG}[\alpha_{PG}^{-1}(\gamma_s^{IK}(w))](q_G) > 0$ for some $q_G \in Q_{PG}$ then there exists $i_q \in I$ and $p \in R_\Gamma^{-1}(q_G)$ with

$$Z_P[\alpha_P^{-1}(\pi_{i_q s}^{IK}(w))](q) > 0. \tag{52}$$

If $Z_P[\alpha_P^{-1}(\pi_{is}^{IK}(w))](p) > 0$ for some $p \in Q_P \setminus (E_\Gamma \cup P_\Gamma)$ and $i \in I$ then there exists $p_G \in R_\Gamma(p)$ with

$$Z_{PG}[\alpha_{PG}^{-1}(\gamma_s^{IK}(w))](p_G) > 0. \tag{53}$$

For the following definition let $\mathcal{F}_P$ resp. $\mathcal{F}_{PF}$ resp. $\mathcal{F}_{PG}$ be the set of final states of the automaton $\mathbb{P}$ resp. $\mathbb{PF}$ resp. $\mathbb{PG}$.

Definition 15. *A state of $q_F \in Q_{PF}$ has the* completion property *iff for each $x \in pre(\pi_\Phi(P^\sqcup)) \cap SF$ with $Z_{PF}[\alpha_{PF}^{-1}(x)](q_F) > 0$ and each $(y, q) \in (pre(P^\sqcup) \cap L) \times R_\Phi^{-1}(q_F)$ with $Z_P[\alpha_P^{-1}(y)](q) > 0$ it holds $q \notin E_\Gamma$ and if $q \in P_\Gamma$ then there exists $y' \in y^{-1}(L) \cap x^{-1}(SF) \cap \Phi^+$ with $\delta_P(q, y') \in \mathcal{F}_P$ and $\delta_{PF}(q_F, y') \in \mathcal{F}_{PF}$ and if $q \notin P_\Gamma$, then for each $(z, q_G) \in (pre(\pi_\Gamma(P^\sqcup)) \cap SG) \times R_\Gamma(q)$ with $Z_{PG}[\alpha_{PG}^{-1}(z)](q_G) > 0$ there exists $y'' \in y^{-1}(L) \cap \pi_\Phi^{-1}(x^{-1}(SF)) \cap \pi_\Gamma^{-1}(z^{-1}(SG))$ with $\delta_P(q, y'') \in \mathcal{F}_P$, $\delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}$, $\delta_{PG}(q_G, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ and $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.*

In a corresponding manner it is defined how a state $q_G \in Q_{PG}$ has the completion property.

Now we are able to formulate condition III:

Condition III. *For each $u \in pre(\pi_\Phi(P^\sqcup)) \cap SF$ with $\sum_{p \in Q_{PF}} Z_{PF}[\alpha_{PF}^{-1}(u)](p) > 1$ there exits $q_F \in Q_{PF}$ with $Z_{PF}[\alpha_{PF}^{-1}(u)](q_F) > 0$, which has the completion property and for each $v \in pre(\pi_\Gamma(P^\sqcup)) \cap SG$ with $\sum_{p \in Q_{PG}} Z_{PG}[\alpha_{PG}^{-1}(v)](p) > 1$ there exits $q_G \in Q_{PG}$ with $Z_{PG}[\alpha_{PG}^{-1}(v)](q_G) > 0$, which has the completion property.*

To check the completion property for a state $q_F \in Q_{PF}$ first of all the sets R_Φ, E_Γ and P_Γ have to be determined. This can be done by constructing the product automaton of $\mathbb{P}$ and $\mathbb{PF}$ resp, $\mathbb{P}$ and $\mathbb{PG}$ whose state set is R_Φ resp. R_Γ.

In the definition of the completion property there are quantifications over $x \in \text{pre}(\pi_\Phi(P^\sqcup)) \cap SF$, $y \in \text{pre}(P^\sqcup) \cap L$ and $z \in \text{pre}(\pi_\Gamma(P^\sqcup)) \cap SG$.

As x, y and z only appear in the terms $Z_{PF}[\alpha_{PF}^{-1}(x)]$, $x^{-1}(SF)$, $Z_P[\alpha_P^{-1}(y)]$, $y^{-1}(L)$, $Z_{PG}[\alpha_{PG}^{-1}(z)]$ and $z^{-1}(SG)$, which are the state components of the product automaton of $\mathbb{SF}$ and $\mathbb{PF}^\sqcup$, resp. $\mathbb{L}$ and $\mathbb{P}^\sqcup$, resp. $\mathbb{SG}$ and $\mathbb{PG}^\sqcup$ these quantification can be checked by inspecting these product automata.

(Note that the left quotients of a formal language can be identified with the states of its minimal automaton [Sakarovitch 2009])

By the same argument condition III can be checked by inspecting the product automaton of $\mathbb{SF}$ and $\mathbb{PF}^\sqcup$ as well as the product automaton of $\mathbb{SG}$ and $\mathbb{PG}^\sqcup$.

We demonstrate this in our example 2 and prove condition III:

The second components of the states in Fig. 11(b) show that
$\sum_{p \in Q_{PF}} Z_{PF}[\alpha_{PF}^{-1}(u)](p) \leq 1$ for each $u \in \text{pre}(\pi_\Phi(P^{\sqcup})) \cap SF$.

So no $q_F \in Q_{PF}$ with the completion property has to be found.

The second components of the states in Fig. 12(b) show that
$\sum_{p \in Q_{PG}} Z_{PG}[\alpha_{PG}^{-1}(v)](p) > 1$ holds only for those $v \in \text{pre}(\pi_\Gamma(P^{\sqcup})) \cap SG$ with
$Z_{PG}[\alpha_{PG}^{-1}(v)] = \{(\text{III}, 1), (\text{II}, 1)\}$.

So for these v $q_G \in Q_{PG}$ with $Z_{PG}[\alpha_{PG}^{-1}(v)](q_G) > 0$, which has the completion property, has to be found.

So $q_G \in \{\text{III}, \text{II}\} \subset Q_{PG}$.

Now we show that $\text{III} \in Q_{PG}$ has the completion property. For that purpose the sets R_Γ, E_Φ and P_Φ are needed.

The product automaton of $\mathbb{P}$ and $\mathbb{PG}$ accepting $\text{pre}(P) \cap \pi_\Gamma^{-1}(\pi_\Gamma(\text{pre}(P))) = \text{pre}(P)$ is given in Fig. 13.

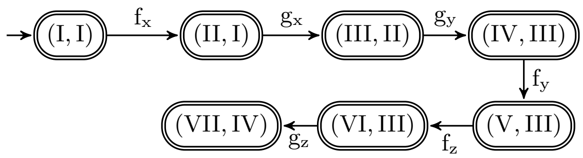


Fig. 13. Product automaton of $\mathbb{P}$ and $\mathbb{PG}$

So

$$R_\Gamma = \{(\text{I}, \text{I}), (\text{II}, \text{I}), (\text{III}, \text{II}), (\text{IV}, \text{III}), (\text{V}, \text{III}), (\text{VI}, \text{III}), (\text{VII}, \text{IV})\} \subset Q_P \times Q_{PG}. \tag{54}$$

To determine E_Φ and P_Φ we first compute the product automaton of $\mathbb{P}$ and $\mathbb{PF}$ accepting $\text{pre}(P) \cap \pi_\Phi^{-1}(\pi_\Phi(\text{pre}(P))) = \text{pre}(P)$. Fig. 14 shows this automaton.

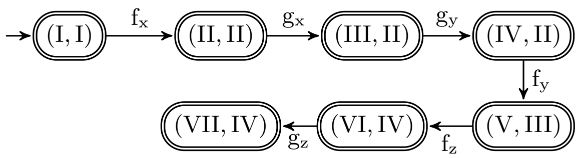


Fig. 14. Product automaton of $\mathbb{P}$ and $\mathbb{PF}$

Fig. 14 shows

$$E_\Phi = \emptyset \text{ and } P_\Phi = \{\text{VI}\}. \tag{55}$$

By definition 15 the state $q_G = \text{III} \in Q_{PG}$ has the completion property iff
for each $z \in \text{pre}(\pi_\Gamma(P^{\sqcup})) \cap SG$ with $Z_{PG}[\alpha_{PG}^{-1}(z)](\text{III}) > 0$ and
each $(y, q) \in (\text{pre}(P^{\sqcup}) \cap L) \times R_\Gamma^{-1}(\text{III})$ with $Z_P[\alpha_P^{-1}(y)](q) > 0$

it holds $q \notin E_\Phi$ and
if $q \in P_\Phi$ then there exists $y' \in y^{-1}(L) \cap z^{-1}(SG) \cap \Gamma^+$ with
$\delta_P(q, y') \in \mathcal{F}_P$ and $\delta_{PG}(\mathrm{III}, y') \in \mathcal{F}_{PG}$, and
if $q \notin P_\Phi$, then for each $(x, q_F) \in (\mathrm{pre}(\pi_\Phi(P^{\sqcup\sqcup})) \cap SF) \times R_\Phi(q)$
with $Z_{PF}[\alpha_{PF}^{-1}(x)](q_F) > 0$ there exists
$y'' \in y^{-1}(L) \cap \pi_\Phi^{-1}(x^{-1}(SF)) \cap \pi_\Gamma^{-1}(z^{-1}(SG))$ with $\delta_P(q, y'') \in \mathcal{F}_P$, $\delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}$,
$\delta_{PG}(\mathrm{III}, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ and
$\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

For $\mathrm{III} \in Q_{PG}$ (54) implies

$$R_\Gamma^{-1} = \{\mathrm{IV}, \mathrm{V}, \mathrm{VI}\} \subset Q_P. \tag{56}$$

Now $z \in \mathrm{pre}(\pi_\Gamma(P^{\sqcup\sqcup})) \cap SG$ and $Z_{PG}[\alpha_{PG}^{-1}(z)](\mathrm{III}) > 0$ implies by the automaton in Fig. 12(b)

$$\delta_{SG}(1, z) \in \{3, 4\}, \tag{57}$$

where δ_{SG} is the state transition function of the automaton $\mathbb{SG}$ in Fig. 5(c).

$(y, q) \in (\mathrm{pre}(P^{\sqcup\sqcup}) \cap L) \times R_\Gamma^{-1}(\mathrm{III})$ and $Z_P[\alpha_P^{-1}(y)](q) > 0$ implies by (56) and by the automaton in Fig. 10

$$q = \mathrm{IV} \in Q_P \text{ and } \delta_L(1, y) = 4 \text{ or} \tag{58a}$$
$$q = \mathrm{V} \in Q_P \text{ and } \delta_L(1, y) = 5 \text{ or} \tag{58b}$$
$$q = \mathrm{VI} \in Q_P \text{ and } \delta_L(1, y) \in \{6, 7, 8\}, \tag{58c}$$

where δ_L is the state transition function of the automaton $\mathbb{L}$ in Fig. 5(a). In each of these cases holds $q \in E_\Phi$ on account of (55).

Also by (55) $q \in P_\Phi$ for case (58c). In that case $\delta_P(q, \mathrm{g_z}) = \delta_P(\mathrm{VI}, \mathrm{g_z}) \in \mathcal{F}_P$ and $\delta_{PG}(\mathrm{III}, \mathrm{g_z}) \in \mathcal{F}_{PG}$.

By (57) the automaton $\mathbb{SG}$ implies $\mathrm{g_z} \in z^{-1}(SG)$, and by (58c) the automaton $\mathbb{L}$ implies $\mathrm{g_z} \in y^{-1}(L)$. So $y' := \mathrm{g_z} \in y^{-1}(L) \cap z^{-1}(SG) \cap \Gamma^+$, which implies the completion property of $\mathrm{III} \in Q_{PG}$ in case (58c).

If $q \in \{\mathrm{IV}, \mathrm{V}\}$ then by (55) $q \notin P_\Phi$. So we need the relation $R_\Phi \subset Q_P \times Q_{PF}$.

By the product automaton of Fig. 14

$$R_\Phi = \{(\mathrm{I}, \mathrm{I}), (\mathrm{II}, \mathrm{II}), (\mathrm{III}, \mathrm{II}), (\mathrm{IV}, \mathrm{II}), (\mathrm{V}, \mathrm{III}), (\mathrm{VI}, \mathrm{IV}), (\mathrm{VII}, \mathrm{IV})\}. \tag{59}$$

Now in case (58a) $(x, q_F) \in (\mathrm{pre}(\pi_\Phi(P^{\sqcup\sqcup})) \cap SF) \times R_\Phi(q)$ and $Z_{PF}[\alpha_{PF}^{-1}(x)](q_F) > 0$ implies by (59) and by the automaton in Fig. 11(b)

$$q_F = \mathrm{II} \in Q_{PF} \text{ and } \delta_{SF}(1, x) = 2, \tag{60}$$

where δ_{SF} is the state transition function of the automaton $\mathbb{SF}$ in Fig.5(b).

By (58a) the automaton $\mathbb{P}$ implies $\delta_P(q, \mathrm{f_y f_z g_z}) = \delta_P(\mathrm{IV}, \mathrm{f_y f_z g_z}) \in \mathcal{F}_P$, and the automaton $\mathbb{L}$ implies $\mathrm{f_y f_z g_z} \in y^{-1}(L)$.

By (60) the automaton $\mathbb{PF}$ implies $\delta_{PF}(q_F, \mathrm{f_y f_z}) = \delta_{PF}(\mathrm{II}, \mathrm{f_y f_z}) \in \mathcal{F}_{PF}$, and the automaton $\mathbb{SF}$ implies $\mathrm{f_y f_z} \in x^{-1}(SF)$.

The automaton $\mathbb{PG}$ implies $\delta_{PG}(\mathrm{III}, \mathrm{g_z}) \in \mathcal{F}_{PG}$, and by (57) the automaton $\mathbb{SG}$ implies $\mathrm{g_z} \in z^{-1}(SG)$.

So $y'' := \mathrm{f_y f_z g_z}$ fulfills the conditions of the completion property of $\mathrm{III} \in Q_{PG}$ in case (58a), because of $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

By a corresponding argument in case (58b)

$$q_F = \text{III} \in Q_{PF} \text{ and } \delta_{SF}(1, x) = 3. \tag{61}$$

This implies that $y'' := \mathrm{f_z g_z}$ fulfills the necessary conditions to complete the proof of the completion property of $\text{III} \in Q_{PG}$.

So our example 2 fulfills condition III.

Using (50) - (53) we are able to prove

Theorem 9. *Let condition I - condition III be fulfilled and* $w \in \mathcal{L}_{IK}$.

(9a) If $\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) > 1$ *for* $r \in I$
then there exists $s \in K$ *and* $w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+$ *with*

$$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) < \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)),$$
$$\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) < \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) \text{ and}$$
$$\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) \leq \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)).$$

(9b) If $\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)) > 1$ *for* $s \in K$
then there exists $r \in I$ *and* $w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+$ *with*

$$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) < \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)),$$
$$\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) < \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)) \text{ and}$$
$$\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) \leq \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)).$$

Proof. The proof of (9b) is analogue to (9a), so it is sufficient to prove (9a).

$w \in \mathcal{L}_{IK}$ implies $\varphi_r^{IK}(w) \in SF$ and hence by condition II $\varphi_r^{IK}(w) \in SF \cap \mathrm{pre}((\pi_\Phi(P))^{\sqcup})$ and $\mathbb{PF}$ is deterministic on $\varphi_r^{IK}(w)$.

So by definition $\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) = \sum_{q \in Q_{PF}} [Z_{PF}(\alpha_{PF}^{-1}(\varphi_r^{IK}(w)))](q)$.

Now on account of condition III and $\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) > 1$ there exists $q_F \in Q_{PF}$ with

$$Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))](q_F) > 0, \tag{62}$$

which has the completion property.

By (50) there exist $k_q \in K$ and $q \in R_\Phi^{-1}(q_F)$ with

$$Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))](q) > 0. \tag{63}$$

Now condition I, condition II and the completion property of q_F imply (with $y = \pi_{rk_q}^{IK}(w)$ and $x = \varphi_r^{IK}(w)$) $q \notin E_\Gamma$ and for each of the two cases $q \in P_\Gamma$ or $q \notin P_\Gamma$ the existence of certain continuations of $\pi_{rk_q}^{IK}(w)$ in L.

Case 1: $q \in P_\Gamma$

By the completion property of q_F there exits

$$y' \in (\pi_{rk_q}^{IK}(w))^{-1}(L) \cap (\varphi_r^{IK}(w))^{-1}(SF) \cap \Phi^+ \tag{64}$$

with $\delta_P(q, y') \in \mathcal{F}_P$ and $\delta_{PF}(q_F, y') \in \mathcal{F}_{PF}$. We now show that $s := k_q \in K$ and $w' := (\pi_{rk_q}^{\{r\}\{k_q\}})^{-1}(y') \in \Phi_{rk_q}^+$ fulfill (9a). (Note that $\pi_{rk_q}^{\{r\}\{k_q\}} : \Sigma_{rk_q}^* \to \Sigma^*$ is an isomorphism.)

For this w' it holds $\pi_{rk_q}^{IK}(w') = y'$, $\varphi_r^{IK}(w') = y'$, $\pi_{ik}^{IK}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r,k_q)\}$, $\varphi_i^{IK}(w') = \varepsilon$ for $i \in I \setminus \{r\}$ and $\gamma_k^{IK}(w') = \varepsilon$ for $k \in K$.

Together with (64) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^{+}. \tag{65}$$

Let $y' = a_1 \dots a_n$ with $n \geq 1$ and $a_i \in \Phi$ for $i \in \{1,\dots,n\}$. On account of $\delta_P(q,y') \in \mathcal{F}_P$ for $i \in \{1,\dots,n+1\}$ there exists $q_i \in Q_P$ with $q = q_1$, $\delta_P(q_i,a_i) = q_{i+1}$ for $i \in \{1,\dots,n\}$ and $q_{n+1} \in \mathcal{F}_P$.

According to the definition of $\mathbb{P}^{\sqcup\!\sqcup}$ let

$$\Delta_P^{\sqcup\!\sqcup} = \tilde{\Delta}_P \cup \mathring{\Delta}_P \cup \bar{\Delta}_P \cup \tilde{\bar{\Delta}}_P \subset \mathbb{N}_0^{Q_P} \times \Sigma \times \mathbb{N}_0^{Q_P}$$

be the state transition relation of $\mathbb{P}^{\sqcup\!\sqcup}$ and $A_P \subset (\Delta_P^{\sqcup\!\sqcup})^*$ the set of all possible paths in $\mathbb{P}^{\sqcup\!\sqcup}$ starting with the initial state 0 and including the empty path ε.

Let $f_1 = Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))] \in \mathbb{N}_0^{Q_P}$.

By (63) $f_1 \geq 1_q = 1_{q_1}$.

For $i \in \{1,\dots,n-1\}$ let $f_{i+1} := f_i - 1_{q_i} + 1_{q_i+1}$ and let $f_{n+1} := f_n - 1_{q_n}$.

This implies $f_i \geq 1_{q_i}$ and

$$\sum_{p \in Q_P} f_i(p) = \sum_{p \in Q_P} f_1(p) \text{ for } i \in \{1,\dots,n\} \text{ and}$$
$$\sum_{p \in Q_P} f_{n+1}(p) = \sum_{p \in Q_P} f_1(p) - 1. \tag{66}$$

By definition of $\Delta_P^{\sqcup\!\sqcup}$

$$(f_i, a_i, f_{i+1}) \in \mathring{\Delta}_P \text{ and } (f_n, a_n, f_{n+1}) \in \bar{\Delta}_P.$$

Hence

$$\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))(f_1,a_1,f_2)\dots(f_n,a_n,f_{n+1}) \in A_P.$$

This implies

$$\alpha_P^{-1}(\pi_{rk_q}^{IK}(ww')) = \alpha_P^{-1}(\pi_{rk_q}^{IK}(w))(f_1,a_1,f_2)\dots(f_n,a_n,f_{n+1})$$

because by (65) and condition I $\mathbb{P}^{\sqcup\!\sqcup}$ is deterministic on $\pi_{rk_q}^{IK}(ww')$.

So by (66) we get
$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) = \mathcal{D}_{\mathbb{P}}(\pi_{rk_q}^{IK}(ww')) = \sum_{p \in Q_P} Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(ww'))](p) = \sum_{p \in Q_P} f_{n+1}(p) = \sum_{p \in Q_P} f_1(p) - 1 = \sum_{p \in Q_P} Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))](p) - 1 = \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)) - 1.$

The same argumentation concerning $\mathbb{PF}^{\sqcup\!\sqcup}$ shows

$$\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) = \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) - 1.$$

Because of $w' \in \Phi_{rs}^{+}$ it holds $\gamma_s^{IK}(w') = \varepsilon$ and therefore

$$\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) = \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)),$$

which completes the proof of (9a) for case (1).

Case 2: $q \notin P_\Gamma$
Now by (63) and (53) there exists $q_G \in R_\Gamma(p)$ with $Z_{PG}[\alpha_{PG}^{-1}(\gamma_{k_q}^{IK}(w))](q_G) > 0$.

So condition II and the completion property of q_F imply (with $z = \gamma^{IK}_{k_q}(w)$) the existence of

$$y'' \in (\pi^{IK}_{rk_q}(w))^{-1}(L) \cap \pi_\Phi^{-1}((\varphi^{IK}_r(w))^{-1}(SF)) \\ \cap \pi_\Gamma^{-1}((\gamma^{IK}_{k_q}(w))^{-1}(SG)) \qquad (67)$$

with $\delta_P(q, y'') \in \mathcal{F}_P$ and $\delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}$, $\delta_{PG}(q_G, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ and $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

We now show that $s := k_q \in K$ and $w' := (\pi^{\{r\}\{k_q\}}_{rk_q})^{-1}(y'') \in \Sigma^+_{rk_q}$ fulfill (9a). For this w' it holds $\pi^{IK}_{rk_q}(w') = y''$, $\varphi^{IK}_r(w') = \pi_\Phi(y'')$, $\gamma^{IK}_{k_q}(w') = \pi_\Gamma(y'')$, $\pi^{IK}_{ik}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r, k_q)\}$, $\varphi^{IK}_i(w') = \varepsilon$ for $i \in I \setminus \{r\}$ and $\gamma^{IK}_k(w') = \varepsilon$ for $k \in K \setminus \{k_q\}$.

Together with (67) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma^+_{rs}. \qquad (68)$$

Now the same argumentation as in case 1 shows

$$\mathcal{D}_{\mathbb{P}}(\pi^{IK}_{rs}(ww')) = \mathcal{D}_{\mathbb{P}}(\pi^{IK}_{rs}(w)) - 1,$$
$$\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_r(ww')) = \mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_r(w)) - 1 \text{ and}$$
$$\mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_s(ww')) = \mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_s(w)) - 1,$$

which completes the proof of theorem 9. □

Iteration of theorem 9 proves

Corollary 2. *Let condition I - condition III be fulfilled, then for each $w \in \mathcal{L}_{IK}$ there exists $\hat{w} \in w^{-1}(\mathcal{L}_{IK})$ such that*

$$\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_i(w\hat{w})) \leq 1, \mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_k(w\hat{w})) \leq 1 \textit{ and } \mathcal{D}_{\mathbb{P}}(\pi^{IK}_{ik}(w\hat{w})) \leq \mathcal{D}_{\mathbb{P}}(\pi^{IK}_{ik}(w))$$

for each $i \in I$ and $k \in K$.

By (49) $\sum_{q \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi^{IK}_{rk}(w))](q) > 1$ implies the existence of $t, t' \in \mathbb{N}$ with $t \neq t'$, $\delta_P(q_{P0}, \tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t)}(w'')) \in Q_P \setminus (E_\Phi \cup P_\Phi)$, $\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t)}(w'') \notin P \cup \{\varepsilon\}$, $\delta_P(q_{P0}, \tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t')}(w'')) \in Q_P \setminus (E_\Phi \cup P_\Phi)$ and $\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t')}(w'') \notin P \cup \{\varepsilon\}$.

By the definition of E_Φ and P_Φ this implies
$\delta_{PF}(q_{PF0}, \pi_\Phi(\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t)}(w''))) \in Q_{PF}$, $\pi_\Phi(\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t)}(w'')) \notin \pi_\Phi(P) \cup \{\varepsilon\}$,
$\delta_{PF}(q_{PF0}, \pi_\Phi(\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t')}(w''))) \in Q_{PF}$ and $\pi_\Phi(\tau^{\{r\}\times K \times \mathbb{N}}_{(r,k,t')}(w'')) \notin \pi_\Phi(P) \cup \{\varepsilon\}$.
Hence by (48)

$$\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_r(w)) = \sum_{q_F \in Q_{PF}} Z_{PF}[\alpha_{PF}^{-1}(\varphi^{IK}_r(w))](q_F) > 1.$$

By analogous argumentation

$$\sum_{q \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi^{IK}_{rk}(w))](q) > 1$$

implies $\mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_k(w)) > 1$ for $(r,k) \in I \times K$.

So corollary 2 implies

Corollary 3.

$$\sum_{q\in Q_P\setminus(E_\Phi\cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w\hat{w}))](q) \leq 1 \textit{ and}$$
$$\sum_{q\in Q_P\setminus(E_\Gamma\cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w\hat{w}))](q) \leq 1$$

for $(i,k) \in I \times K$.

Condition IV. *For each* $z \in pre(\pi_\Gamma(P^{\sqcup\!\sqcup})) \cap SG$ *and* $q_G \in Q_{PG}$ *with* $Z_{PG}[\alpha_{PG}^{-1}(z)] = 1_{q_G}$ *and each* $(y,q) \in (pre(P^{\sqcup\!\sqcup}) \cap L) \times R_\Gamma^{-1}(q_G)$ *with* $Z_P[\alpha_P^{-1}(y)](q) > 0$, $\sum_{q'\in Q_P\setminus(E_\Phi\cup P_\Phi)} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ *and* $\sum_{q'\in Q_P\setminus(E_\Gamma\cup P_\Gamma)} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ *it holds*
$q \notin E_\Phi$ *and*
if $q \in P_\Phi$ *then there exists* $y' \in y^{-1}(L) \cap z^{-1}(SG) \cap \Gamma^+$ *with* $\delta_P(q,y') \in \mathcal{F}_P$ *and* $\delta_{PG}(q_G,y') \in \mathcal{F}_{PG}$, *and*
if $q \notin P_\Phi$ *then for each* $(x,q_F) \in (pre(\pi_\Phi(P^{\sqcup\!\sqcup})) \cap SF) \times R_\Phi(q)$ *with* $Z_{PF}[\alpha_{PF}^{-1}(x)] = 1_{q_F}$ *there exists* $y'' \in y^{-1}(L) \cap \pi_\Phi^{-1}(x^{-1}(SF)) \cap \pi_\Gamma^{-1}(z^{-1}(SG))$ *with* $\delta_P(q,y'') \in \mathcal{F}_P$, $\delta_{PF}(q_F,\pi_\Phi(y'')) \in \mathcal{F}_{PF}$, $\delta_{PG}(q_G,\pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ *and* $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

We now show that example 2 fulfills condition IV.

The states of the automaton in Fig. 12(b) show that $Z_{PG}[\alpha_{PG}^{-1}(z)] = 1_{q_G}$ holds only for those $z \in \text{pre}(\pi_\Gamma(P^{\sqcup\!\sqcup})) \cap SG$ and $q_G \in Q_{PG}$ with

$$q_G = \text{II and } \delta_{SG}(1,z) = 2 \text{ or} \tag{69a}$$
$$q_G = \text{III and } \delta_{SG}(1,z) = 3. \tag{69b}$$

Now (54) implies

$$R_\Gamma^{-1}(\text{II}) = \{\text{III}\} \subset Q_P \text{ and} \tag{70a}$$
$$R_\Gamma^{-1}(\text{III}) = \{\text{IV}, \text{V}, \text{VI}\} \subset Q_P. \tag{70b}$$

Fig. 13 shows

$$E_\Gamma = \{\text{II}\} \text{ and } P_\Gamma = \emptyset \tag{71}$$

By (70a), Fig. 10, (55) and (71)
$(y,q) \in (\text{pre}(P^{\sqcup\!\sqcup}) \cap L) \times R_\Gamma^{-1}(\text{II})$ with $Z_P[\alpha_P^{-1}(y)](q) > 0$, $\sum_{q'\in Q_P\setminus(E_\Phi\cup P_\Phi)} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ and $\sum_{q'\in Q_P\setminus(E_\Gamma\cup P_\Gamma)} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ implies

$$q = \text{III} \in Q_p \text{ and } \delta_L(1,y) = 3. \tag{72}$$

By (70b), Fig. 10, (55) and (71)
$(y,q) \in (\text{pre}(P^{\sqcup\!\sqcup}) \cap L) \times R_\Gamma^{-1}(\text{III})$ with $Z_P[\alpha_P^{-1}(y)](q) > 0$,

$$\sum_{q' \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(y)](q') \leq 1 \text{ and } \sum_{q' \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(y)](q') \leq 1 \text{ implies}$$

$$q = \text{IV} \in Q_p \text{ and } \delta_L(1, y) = 4 \text{ or} \tag{73a}$$

$$q = \text{V} \in Q_p \text{ and } \delta_L(1, y) = 5 \text{ or} \tag{73b}$$

$$q = \text{VI} \in Q_p \text{ and } \delta_L(1, y) \in \{6, 7\}. \tag{73c}$$

On account of (55) in each of these 4 cases holds $q \notin E_\Phi$.

Also by (55) $q \in P_\Phi$ iff $q = \text{VI} \in Q_P$, which is case (73c). Hence $q_G = \text{III} \in Q_{PG}$, $\delta_{SG}(1, z) = 3$ and $\delta_L(1, y) \in \{6, 7\}$. In that case the automata $\mathbb{P}$ (Fig. 6), $\mathbb{PG}$ (Fig. 12(a)), $\mathbb{SG}$ (Fig. 5(c)) and $\mathbb{L}$ (Fig. 5(a)) show $\delta_P(q, \text{g}_\text{z}) = \delta_P(\text{VI}, \text{g}_\text{z}) \in \mathcal{F}_P$, $\delta_{PG}(q_G, \text{g}_\text{z}) = \delta_{PG}(\text{III}, \text{g}_\text{z}) \in \mathcal{F}_{PG}$, $\text{g}_\text{z} \in z^{-1}(SG)$ and $\text{g}_\text{z} \in y^{-1}(L)$.

So $y' := \text{g}_\text{z} \in y^{-1}(L) \cap \Gamma^+ \cap z^{-1}(SG)$ fulfills condition IV in case (73c).

In each of the other 3 cases (72), (73a) and (73b) holds $q \notin P_\Phi$.

In case (72) holds $q_G = \text{II} \in Q_{PG}$, $\delta_{SG}(1, z) = 2$, $q = \text{III} \in Q_P$, $\delta_L(1, y) = 3$ and by (59) $R_\Phi(\text{III}) = \{\text{II}\} \subset Q_{PF}$.

Hence by the automaton in Fig.11(b)
$(x, q_F) \in (\text{pre}(\pi_\Phi(P^{\sqcup})) \cap SF) \times R_\Phi(\text{III})$
with $Z_{PF}[\alpha_{PF}^{-1}(x)] = 1_{q_F}$ implies $q_F = \text{II} \in Q_{PF}$ and $\delta_{SF}(1, x) = 2$.

In that case the automata $\mathbb{P}$ (Fig. 6), $\mathbb{PF}$ (Fig. 11(a)), $\mathbb{PG}$ (Fig. 12(a)), $\mathbb{SF}$ (Fig. 5(b)), $\mathbb{SG}$ (Fig. 5(c)) and $\mathbb{L}$ (Fig. 5(a)) show

$$\begin{aligned} \delta_P(q, \text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) &= \delta_P(\text{III}, \text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) \in \mathcal{F}_P, \\ \delta_{PF}(q_F, \pi_\Phi(\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z})) &= \delta_{PF}(\text{II}, \text{f}_\text{y}\text{f}_\text{z}) \in \mathcal{F}_{PF}, \\ \delta_{PG}(q_G, \pi_\Gamma(\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z})) &= \delta_{PG}(\text{II}, \text{g}_\text{y}\text{g}_\text{z}) \in \mathcal{F}_{PG}, \end{aligned}$$

$\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z} \in y^{-1}(L)$, $\pi_\Phi(\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) \in x^{-1}(SF)$ and $\pi_\Gamma(\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z} \in z^{-1}(SG)$.
So $y'' := \text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}$ fulfills condition IV in case (72).

In case (73a) holds $q_G = \text{III} \in Q_{PG}$, $\delta_{SG}(1, z) = 3$, $q = \text{IV} \in Q_p$, $\delta_L(1, y) = 4$ and by (59) $R_\Phi(\text{IV}) = \{\text{II}\} \subset Q_{PF}$.

In case (73b) holds $q_G = \text{III} \in Q_{PG}$, $\delta_{SG}(1, z) = 3$, $q = \text{V} \in Q_p$, $\delta_L(1, y) = 5$ and by (59) $R_\Phi(\text{V}) = \{\text{III}\} \subset Q_{PF}$.

Now by the same argumentation as in case (72) $y'' := \text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}$ fulfills condition IV in case (73a) and $y'' := \text{f}_\text{z}\text{g}_\text{z}$ fulfills condition IV in case (73b). So example 2 fulfills condition IV.

Theorem 10. *Let condition I - IV be fulfilled and $w \in \mathcal{L}_{IK}$ with $\mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w)) \leq 1$ and $\mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w)) \leq 1$ for $(i, k) \in I \times K$.*
If $\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)) = 1$ for $s \in K$ then there exists $r \in I$ and $w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+$ with

$$\begin{aligned} &\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) < \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)), \\ &\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) = 0 \textit{ and} \\ &\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) \leq \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)). \end{aligned}$$

Proof. By corollary 3

$$\sum_{q' \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w))](q') \leq 1 \text{ and}$$

$$\sum_{q' \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w))](q') \leq 1 \tag{74}$$

for each $(i,k) \in I \times K$.

$\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)) = 1$ for $s \in K$ imply the existence of $q_G \in Q_{PG}$ with

$$Z_{PG}[\alpha_{PG}^{-1}(\gamma_s^{IK}(w))](q') = 1_{q_G} \tag{75}$$

Now by (52) there exists $i_q \in I$ and $q \in R_\Gamma^{-1}(q_G)$ with

$$Z_P[\alpha_P^{-1}(\pi_{i_q s}^{IK}(w))](q) > 0. \tag{76}$$

(74), (75), (76) and condition IV imply (with $z = \gamma_s^{IK}(w)$ and $y = \pi_{i_q s}^{IK}(w)$) $q \notin E_\Phi$ and for each of the two cases $q \in P_\Phi$ or $q \notin P_\Phi$ the existence of certain continuations of $\pi_{i_q s}^{IK}(w)$ in L.

Case(1): $q \in P_\Phi$

By condition IV there exists

$$y' \in (\pi_{i_q s}^{IK}(w))^{-1}(L) \cap (\gamma_s^{IK}(w))^{-1}(SG) \cap \Gamma^+ \tag{77}$$

with $\delta_P(q,y') \in \mathcal{F}_P$ and $\delta_{PG}(q_G,y') \in \mathcal{F}_{PG}$.

We now show that $r := i_q \in I$ and $w' := (\pi_{rs}^{\{r\}\{s\}})^{-1}(y') \in \Gamma_{rs}^+ \subset \Sigma_{rs}^+$ fulfill the statement of theorem 10.

For this w' it holds $\pi_{rs}^{IK}(w') = y'$, $\gamma_s^{IK}(w') = y'$, $\pi_{ik}^{IK}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r,s)\}$, $\gamma_k^{IK}(w') = \varepsilon$ for $k \in K \setminus \{s\}$ and $\varphi_i^{IK}(w') = \varepsilon$ for $i \in I$.

Together with (77) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+ \tag{78}$$

Let $y' = a_1 \ldots a_n$ with $n \geq 1$ and $a_i \in \Gamma$ for $i \in \{1,\ldots,n\}$. On account of $\delta(q,y') \in \mathcal{F}_P$ for $i \in \{1,\ldots,n{+}1\}$ there exists $q_i \in Q_P$ with $q = q_1$, $\delta_P(q_i,a_i) = q_{i+1}$ for $i \in \{1,\ldots,n\}$ and $q_{n+1} \in \mathcal{F}_P$.

By the same argumentation as in case (1) of the proof of theorem 9 we get

$$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) = \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)) - 1.$$

The same argumentation concerning $\mathbb{PG}^{\sqcup}$ shows $\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) = 0$.

Because of $w' \in \Gamma_{rs}^+$ it holds $\varphi_r^{IK}(w') = \varepsilon$ and therefore

$$\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) = \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)),$$

which completes the proof of theorem 10 for case (1).

Case(2): $q \notin P_\Phi$ and therefore $q \notin E_\Phi \cup P_\Phi$.

Now on account of (76) and (51) there exists $q_F \in R_\Phi(q)$ with $Z_{PF}[\alpha_{PF}^{-1}(\varphi_{i_q}^{IK}(w))](q_F) > 0$, which implies $Z_{PF}[\alpha_{PF}^{-1}(\varphi_{i_q}^{IK}(w))] = 1_{q_F}$ by the assumptions of theorem 10.

So (with $x = \varphi^{IK}_{i_q}(w)$) condition IV implies the existence of

$$y'' \in (\pi^{IK}_{i_q s}(w))^{-1}(L) \cap \pi_\Phi^{-1}((\varphi^{IK}_{i_q}(w))^{-1}(SF)) \\ \cap \pi_\Gamma^{-1}((\gamma^{IK}_s(w))^{-1}(SG)) \tag{79}$$

with $\delta_P(q, y'') \in \mathcal{F}_P$, $\delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}$, $\delta_{PG}(q_G, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ and $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

We now show that $r := i_q \in I$ and $w' := (\pi^{\{r\}\{s\}}_{rs})^{-1}(y'') \in \Sigma^+_{rs}$ fulfill the statement of theorem 10.

For this w' it holds $\pi^{IK}_{rs}(w') = y''$, $\varphi^{IK}_r(w') = \pi_\Phi(y'')$, $\gamma^{IK}_s(w') = \pi_\Gamma(y'')$, $\pi^{IK}_{ik}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r,s)\}$, $\varphi^{IK}_i(w') = \varepsilon$ for $i \in I \setminus \{r\}$ and $\gamma^{IK}_k(w') = \varepsilon$ for $k \in K \setminus \{s\}$.

Together with (79) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma^+_{rs}. \tag{80}$$

Now the same argumentation as in case (1) shows

$$\mathcal{D}_{\mathbb{P}}(\pi^{IK}_{rs}(ww')) = \mathcal{D}_{\mathbb{P}}(\pi^{IK}_{rs}(w)) - 1,$$
$$\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_r(ww')) = \mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_r(w)) - 1 \text{ and}$$
$$\mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_s(ww')) = 0,$$

which completes the proof of theorem 10. □

Iteration of theorem 10 and corollary 2 proves

Corollary 4. *Let condition I - condition IV be fulfilled, then for each $w \in \mathcal{L}_{IK}$ there exists $\hat{w} \in w^{-1}(\mathcal{L}_{IK})$ such that*

$$\mathcal{D}_{\mathbb{P}}(\pi^{IK}_{ik}(w\hat{w})) \leq \mathcal{D}_{\mathbb{P}}(\pi^{IK}_{ik}(w)),$$

$$\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_i(w\hat{w})) \leq 1 \text{ and } \mathcal{D}_{\mathbb{PG}}(\gamma^{IK}_k(w\hat{w})) = 0$$

for each $i \in I$ and $k \in K$.

Now corollary 3 and (53) shows

Corollary 5.

$$\sum_{q \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi^{IK}_{ik}(w\hat{w}))](q) \leq 1 \text{ and}$$
$$\sum_{q \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi^{IK}_{ik}(w\hat{w}))](q) = 0$$

for $(i,k) \in I \times K$.

Now we reduce $\mathcal{D}_{\mathbb{PF}}(\varphi^{IK}_i(w\hat{w}))$ in the situation of corollary 4 and 5 by the following

Condition V. *For each $x \in pre(\pi_\Phi(P^{\sqcup\!\sqcup})) \cap SF$ and $q_F \in Q_{PF}$ with $Z_{PF}[\alpha_{PF}^{-1}(x)] = 1_{q_F}$ and $R_\Phi^{-1}(q_F) \cap (E_\Gamma \cup P_\Gamma) \neq \emptyset$ and each $(y, q) \in (pre(P^{\sqcup\!\sqcup}) \cap L) \times (R_\Phi^{-1}(q_F) \cap (E_\Gamma \cup P_\Gamma))$ with*

$Z_P[\alpha_P^{-1}(y)](q) > 0$, $\sum_{q' \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ *and*
$\sum_{q' \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(y)](q') = 0$ *the following holds:*
If $q \in P_\Gamma$ *then there exists* $y' \in y^{-1}(L) \cap x^{-1}(SF) \cap \Phi^+$ *with* $\delta_P(q, y') \in \mathcal{F}_P$ *and* $\delta_{PF}(q_F, y') \in \mathcal{F}_{PF}$, *and*
if $q \in E_\Gamma \setminus P_\Gamma$, *then for each* $z \in pre(\pi_\Gamma(P^{\sqcup})) \cap SG$ *with* $Z_{PG}[\alpha_{PG}^{-1}(z)] = 0$ *there exists* $y'' \in y^{-1}(L) \cap \pi_\Phi^{-1}(x^{-1}(SF)) \cap \pi_\Gamma^{-1}(z^{-1}(SG))$ *with* $\delta_P(q, y'') \in \mathcal{F}_P$, $\delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}$, $\delta_{PG}(q_{G0}, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$ *and* $\pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y'')$.

We now show that example 2 fulfills condition V.

By (71) $E_\Gamma \cup P_\Gamma = \{\text{II}\} \subset Q_P$.
By definition $\text{II} \in R_\Phi^{-1}(q_F)$ iff $(\text{II}, q_F) \in R_\Phi$.
So by (59)

$$R_\Phi^{-1}(q_F) \cap (E_\Gamma \cup P_\Gamma) \neq \emptyset \text{ iff } q_F = \text{II} \in Q_{PF}. \tag{81}$$

Now the automaton in Fig. 11(b) shows that

$$Z_{PF}[\alpha_{PF}^{-1}(x)] = 1_{\text{II}} \text{ for } x \in \text{pre}(\pi_\Phi(P^{\sqcup})) \cap SF \text{ iff } \delta_{SF}(1, x) = 2. \tag{82}$$

By (55) $E_\Phi \cup P_\Phi = \{\text{VI}\} \subset Q_P$. So by the automaton in Fig. 10 $(y, q) \in (\text{pre}(P^{\sqcup}) \cap L) \times (R_\Phi^{-1}(\text{II}) \cap (E_\Gamma \cup P_\Gamma))$ with
$Z_P[\alpha_P^{-1}(y)](q) > 0$, $\sum_{q' \in Q_P \setminus \{\text{VI}\}} Z_P[\alpha_P^{-1}(y)](q') \leq 1$ and
$\sum_{q' \in Q_P \setminus \{\text{II}\}} Z_P[\alpha_P^{-1}(y)](q') = 0$ implies

$$q = \text{II} \in Q_P \text{ and } \delta_L(1, y) = 2. \tag{83}$$

By (71) $E_\Gamma \setminus P_\Gamma = \{\text{II}\}$ and therefore $q = \text{II} \in E_\Gamma \setminus P_\Gamma$.

The automaton in Fig. 12(b) shows, that

$$Z_{PG}[\alpha_{PG}^{-1}(z)] = 0 \text{ for } z \in \text{pre}(\pi_\Gamma(P^{\sqcup})) \cap SG \text{ iff } \delta_{SG}(1, z) = 1. \tag{84}$$

Now (81) - (84) and the automata $\mathbb{P}$ (Fig. 6), $\mathbb{PF}$ (Fig. 11(a)), $\mathbb{PG}$ (Fig. 12(a)), $\mathbb{SF}$ (Fig. 5(b)), $\mathbb{SG}$ (Fig. 5(c)) and $\mathbb{L}$ (Fig. 5(a)) show

$$\delta_P(q, \text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) = \delta_P(\text{II}, \text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) \in \mathcal{F}_P,$$
$$\delta_{PF}(q_F, \pi_\Phi(\text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z})) = \delta_{PF}(\text{II}, \text{f}_\text{y}\text{f}_\text{z}) \in \mathcal{F}_{PF},$$
$$\delta_{PG}(q_{G0}, \pi_\Gamma(\text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z})) = \delta_{PG}(\text{I}, \text{g}_\text{x}\text{g}_\text{y}\text{g}_\text{z}) \in \mathcal{F}_{PG},$$

$\text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z} \in y^{-1}(L)$, $\pi_\Phi(\text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}) \in x^{-1}(SF)$ and $\pi_\Gamma(\text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z} \in z^{-1}(SG)$.
So $y'' := \text{g}_\text{x}\text{g}_\text{y}\text{f}_\text{y}\text{f}_\text{z}\text{g}_\text{z}$ fulfills condition V, which shows that example 2 fulfills condition V.

Theorem 11. *Let conditions I - V be fulfilled and* $w \in \mathcal{L}_{IK}$ *with* $\mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w)) \leq 1$ *and* $\mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w)) = 0$ *for each* $(i, k) \in I \times K$.

If $\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) = 1$ *for* $r \in I$*, then there exists* $s \in K$ *and* $w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+$ *with*

$$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) < \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)) \text{ and } \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) = 0 = \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')).$$

Proof. By corollary 5

$$\sum_{q' \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w))](q') \leq 1 \text{ and}$$
$$\sum_{q' \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w))](q') = 0 \tag{85}$$

for each $(i,k) \in I \times K$.

$\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) = 1$ for $r \in I$ imply the existence of $q_F \in Q_{PF}$ with

$$Z_{PF}[\alpha_{PF}^{-1}(\varphi_r^{IK}(w))] = 1_{q_F}. \tag{86}$$

Now by (50) there exists $k_q \in K$ and $q \in R_\Phi^{-1}(q_F)$ with

$$Z_P[\alpha_P^{-1}(\pi_{rk_q}^{IK}(w))](q) > 0. \tag{87}$$

On account of (85) this imply

$$q \in R_\Phi^{-1}(q_F) \cap (E_\Gamma \cup P_\Gamma). \tag{88}$$

Now (85) - (88) and condition V imply (with $x = \varphi_r^{IK}(w)$ and $y = \pi_{rk_q}^{IK}(w)$) for each of the two cases $q \in P_\Gamma$ or $q \in E_\Gamma \setminus P_\Gamma$ the existence of certain continuations of $\pi_{rk_q}^{IK}(w)$ in L.

Case (1): $q \in P_\Gamma$

By condition V there exists $y' \in (\pi_{rk_q}^{IK}(w))^{-1}(L) \cap (\varphi_r^{IK}(w))^{-1}(SF) \cap \Phi^+$ with

$$\delta_P(q, y') \in \mathcal{F}_P \text{ and } \delta_{PF}(q_F, y') \in \mathcal{F}_{PF}. \tag{89}$$

We now show that $s := k_q \in K$ and $w' := (\pi_{rs}^{\{r\}\{s\}})^{-1}(y') \in \Phi_{rs}^+ \subset \Sigma_{rs}^+$ fulfill the statement of theorem 11.

For this w' it holds $\pi_{rs}^{IK}(w') = y'$, $\varphi_r^{IK}(w') = y'$, $\pi_{ik}^{IK}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r,s)\}$, $\varphi_i^{IK}(w') = \varepsilon$ for $i \in I \setminus \{r\}$ and $\gamma_k^{IK}(w') = \varepsilon$ for $k \in K$. Together with (89) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+. \tag{90}$$

Let $y' = a_1 \dots a_n$ with $n \geq 1$ and $a_i \in \Phi$ for $i \in \{1, \dots, n\}$. On account of $\delta_P(q, y') \in \mathcal{F}_P$ for $i \in \{1, \dots, n+1\}$ there exists $q_i \in Q_P$ with $q = q_1$, $\delta_P(q_i, a_i) = a_{i+1}$ for $i \in \{1, \dots, n\}$ and $q_{n+1} \in \mathcal{F}_P$.

By the same argumentation as in case (1) of the proof of theorem 9 we get

$$\mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(ww')) = \mathcal{D}_{\mathbb{P}}(\pi_{rs}^{IK}(w)) - 1.$$

The same argumentation concerning $\mathbb{PF}^{\sqcup}$ shows $\mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) = 0$.

Because of $w' \in \Phi_{rs}^+$ it holds $\gamma_s^{IK}(w') = \varepsilon$ and therefore

$$\mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) = \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)),$$

which completes the proof of theorem 11 for case (1).

Case (2): $q \in E_\Gamma \setminus P_\Gamma$.
By assumption of theorem 11 $Z_{PG}[\alpha_{PG}^{-1}(\gamma_{k_q}^{IK}(w))] = 0$.
So (with $z = \gamma_{k_q}^{IK}(w)$) condition V implies the existence of $y'' \in (\pi_{rk_q}^{IK}(w))^{-1}(L) \cap \pi_\Phi^{-1}((\varphi_r^{IK}(w))^{-1}(SF)) \cap \pi_\Gamma^{-1}((\gamma_{k_q}^{IK}(w))^{-1}(SG))$ with

$$\begin{aligned} \delta_P(q, y'') \in \mathcal{F}_P, \delta_{PF}(q_F, \pi_\Phi(y'')) \in \mathcal{F}_{PF}, \\ \delta_{PG}(q_{G0}, \pi_\Gamma(y'')) \in \mathcal{F}_{PG} \text{ and} \\ \pi_\Phi(y'') \neq \varepsilon \neq \pi_\Gamma(y''). \end{aligned} \tag{91}$$

We now show that $s := k_q \in K$ and $w' := (\pi_{rs}^{\{r\}\{s\}})^{-1}(y'') \in \Sigma_{rs}^+$ fulfill the statement of theorem 11.

For this w' it holds $\pi_{rs}^{IK}(w') = y''$, $\varphi_r^{IK}(w') = \pi_\Phi(y'')$, $\gamma_s^{IK}(w') = \pi_\Gamma(y'')$, $\pi_{ik}^{IK}(w') = \varepsilon$ for $(i,k) \in (I \times K) \setminus \{(r,s)\}$, $\varphi_i^{IK}(w') = \varepsilon$ for $i \in I \setminus \{r\}$ and $\gamma_k^{IK}(w') = \varepsilon$ for $k \in K \setminus \{s\}$. Together with (91) this implies

$$w' \in w^{-1}(\mathcal{L}_{IK}) \cap \Sigma_{rs}^+. \tag{92}$$

Now the same argumentation as in case (1) shows,

$$\begin{aligned} \mathcal{D}_\mathbb{P}(\pi_{rs}^{IK}(ww')) = \mathcal{D}_\mathbb{P}(\pi_{rs}^{IK}(w)) - 1, \\ \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(ww')) = \mathcal{D}_{\mathbb{PF}}(\varphi_r^{IK}(w)) - 1 \text{ and} \\ \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(ww')) = \mathcal{D}_{\mathbb{PG}}(\gamma_s^{IK}(w)) \end{aligned}$$

on account of $\delta_{PG}(q_{G0}, \gamma_s^{IK}(w')) = \delta_{PG}(q_{G0}, \pi_\Gamma(y'')) \in \mathcal{F}_{PG}$.
This completes the proof of theorem 11. □

Iteration of theorem 11 and corollary 4 proves

Corollary 6. *Let condition I - V be fulfilled, then for each $w \in \mathcal{L}_{IK}$ there exists $\hat{w} \in w^{-1}(\mathcal{L}_{IK})$ such that*
$\mathcal{D}_\mathbb{P}(\pi_{ik}^{IK}(w\hat{w})) \leq \mathcal{D}_\mathbb{P}(\pi_{ik}^{IK}(w))$ and $\mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w\hat{w})) = 0 = \mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w\hat{w}))$ for each $(i,k) \in I \times K$.

Now (51) and (53) proves

Corollary 7. *For $(i,k) \in I \times K$ holds*

$$\sum_{q \in Q_P \setminus (E_\Phi \cup P_\Phi)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w\hat{w}))](q) = 0 \text{ and}$$

$$\sum_{q \in Q_P \setminus (E_\Gamma \cup P_\Gamma)} Z_P[\alpha_P^{-1}(\pi_{ik}^{IK}(w\hat{w}))](q) = 0.$$

Now $\mathcal{D}_\mathbb{P}(\pi_{ik}^{IK}(w\hat{w})) = 0$ can be deduced from corollary 7 by following

Condition VI.

$$(E_\Phi \cup P_\Phi) \cap (E_\Gamma \cup P_\Gamma) = \emptyset$$

So we get

Corollary 8. *Let condition I - VI be fulfilled, then for each $w \in \mathcal{L}_{IK}$ there exists $\hat{w} \in w^{-1}(\mathcal{L}_{IK})$ such that*

$$\mathcal{D}_\mathbb{P}(\pi_{ik}^{IK}(w\hat{w})) = 0 = \mathcal{D}_{\mathbb{PF}}(\varphi_i^{IK}(w\hat{w})) = \mathcal{D}_{\mathbb{PG}}(\gamma_k^{IK}(w\hat{w}))$$

for each $(i,k) \in I \times K$*, and therefore by theorem 8*

$$w\hat{w} \in \mathcal{L}_{IK} \cap \bigcap_{(i,k)\in I\times K} (\pi^{IK}_{ik})^{-1}(P^{\sqcup\!\sqcup}).$$

For our example we have shown $E_\Phi = \emptyset$ and $P_\Phi = \{\text{VI}\}$ (55) as well as $E_\Gamma = \{\text{II}\}$ and $P_\Gamma = \emptyset$ (71). So condition VI is fulfilled.

Now by theorem 1, theorem 7 and corollary 8 conditions I - VI and self-similarity of $\mathcal{L}_{IK}$ imply simplicity of $\Pi^{IK}_{I'K'}$ on $\mathcal{L}_{IK}$. Therefore by conditions I - VI, together with self-similarity and regularity of $\mathcal{L}_{IK}$, theorem 3 can be used to prove approximate satisfaction of uniformly parameterised behaviour properties.

In Sect. 3 we applied theorem 3 to our example 2, where it remained to prove simplicity of $\Pi^{IK}_{I'K'}$ on $\mathcal{L}_{IK}$. Now this gap is filled by the proofs that example 2 fulfills conditions I - VI.

These proofs show that under certain regularity restrictions (the product automata as in Fig. 10, 11(b) and 12(b) must be finite and deterministic) conditions I - VI can be verified by semi-algorithms based on finite state methods. We only get semi-algorithms but no algorithms, because the product automata are constructed step by step and this procedure does not terminate if the corresponding product automaton is not finite. These semi-algorithms only depend on L, SF, SG and P and don't refer to the general index sets I and K.

Conditions I - VI formalise our strategy to complete phases. There are several, and partly more general, of such completion strategies to prove the statement of corollary 8. The aim of condition I - VI was not only our special set of sufficient conditions for uniformly parameterised behaviour properties but also to demonstrate, how completion of phases strategies and corresponding success conditions can be formalised by deterministic computations in shuffle automata.

6. CONCLUSIONS AND FUTURE WORK

In [Ochsenschläger and Rieke 2011] we have shown in particular that for self-similar parameterised systems $\mathcal{L}_{IK}$ the parameterised problem of verifying a *uniformly parameterised safety property* can be reduced to finite many fixed finite state problems.

Extending this, the main result of the present paper is a finite state verification framework for *uniformly parameterised behaviour properties* capturing the full spectrum of safety and liveness. This uniformly parameterisation exactly fits to the scalability and reliability issues of complex systems or systems of systems such as for example Cloud Computing platforms.

In this framework the concept of structuring cooperations into phases enables completion of phases strategies. Consistent with this, corresponding success conditions are formalised which produce finite state semi-algorithms (independent of the concrete parameter setting) to verify behaviour properties of uniformly parameterised cooperations. The next step should be to integrate these semi-algorithms in our SH verification tool [Ochsenschläger et al. 2000].

Besides safety and liveness properties so called hyperproperties [Clarkson and Schneider 2008] are of interest because they give formalisations for non-interference and non-inference. Further work could be to generalise the approach of this paper to hyperproperties as well as to the Security Modeling Framework (SeMF) approach

[Fuchs et al. 2009], where beside system behaviour also local views of agents and agents knowledge about system behaviour are relevant.

7. APPENDIX

7.1 Proof of theorem 1

Let $h : \Sigma^* \to \Sigma'^*$ be an alphabetic homomorphism, $B \subset \Sigma^*$ prefix closed and $w \in \Sigma^*$. Then

$$\begin{aligned}(h(w))^{-1}(h(B)) =& \{v' \in \Sigma'^* | h(w)v' \in h(B)\}\\ =& \{v' \in \Sigma'^* | \text{ there exists } u \in h^{-1}(\{h(w)\}) \text{ and } v \in u^{-1}(B) \\ & \text{ such that } v' = h(v)\}\\ =& \bigcup_{u \in h^{-1}(\{h(w)\})} \{h(v) \in \Sigma'^* | v \in u^{-1}(B)\}\\ =& \bigcup_{u \in h^{-1}(\{h(w)\})} h(u^{-1}(B)) \supset h(w^{-1}(B)). \end{aligned} \tag{93}$$

Let $x = yz \in \Sigma^*$, then by (93)

$$h(x^{-1}(B)) = h((yz)^{-1}(B)) = h(z^{-1}(y^{-1}(B))) \subset (h(z))^{-1}[h(y^{-1}(B))]. \tag{94}$$

If $h(x^{-1}(B)) = (h(x))^{-1}(h(B))$ then

$$h(x^{-1}(B)) = (h(y)h(z))^{-1}(h(B)) = (h(z))^{-1}[(h(y))^{-1}(h(B))] \text{ and by (94)}$$
$$(h(z))^{-1}[(h(y))^{-1}(h(B))] \subset (h(z))^{-1}[h(y^{-1}(B))]. \tag{95}$$

(93) implies $h(y^{-1}(B)) \subset (h(y))^{-1}(h(B))$ and therefore

$$(h(z))^{-1}[h(y^{-1}(B))] \subset (h(z))^{-1}[(h(y))^{-1}(h(B))]. \tag{96}$$

Now (95) and (96) imply

$$(h(z))^{-1}[h(y^{-1}(B))] = (h(z))^{-1}[(h(y))^{-1}(h(B))]. \tag{97}$$

If $x = yz \in B$ then

$$h(z) \in (h(y))^{-1}(h(B)). \tag{98}$$

(97) and (98) prove theorem 1.

7.2 A sufficient condition for self-similarity

The proof of the following sufficient condition for self-similarity of $\mathcal{L}_{IK}$ is given in [Ochsenschläger and Rieke 2010].

Let $\mathbb{PF} = (\Phi, Q_{PF}, \delta_{PF}, q_{PF0}, F_{PF})$ resp. $\mathbb{PG} = (\Gamma, Q_{PG}, \delta_{PG}, q_{PG0}, F_{PG})$ be deterministic automata that accept PF resp. PG and let $\mathbb{SF} = (\Phi, Q_{SF}, \delta_{SF}, q_{SF0})$ resp. $\mathbb{SG} = (\Gamma, Q_{SG}, \delta_{SG}, q_{SG0})$ be deterministic automata that accept SF resp. SG. If SF is deterministically based on PF w.r.t. $\mathbb{PF}$ resp. SG is deterministically based on PG w.r.t. $\mathbb{PG}$, then holds

Theorem 12. *If for each $(q_{SF}, f) \in Q_{SF} \times \mathbb{N}_0^Q$ and $(q'_{SF}, f') \in Q_{SF} \times \mathbb{N}_0^Q$ for which exists $u, u' \in SF \cap (pre(PF))^{\sqcup}$ such that $q_{SF} = \delta_{SF}(q_{SF0}, u)$, $q'_{SF} = \delta_{SF}(q_{SF0}, u')$,*

$f = Z[\alpha^{-1}(u)]$, $f' = Z[\alpha^{-1}(u')]$ and for which $f \geq f'$ holds

$$\{a \in \Phi \cap pre(PF) \mid \delta_{SF}(q_{SF}, a) \text{ is defined } \} \subset \{a \in \Phi \cap pre(PF) \mid \delta_{SF}(q'_{SF}, a) \text{ is defined } \} \quad (99a)$$

and for each $q_{PF} \in Q_{PF}$ with $f'(q_{PF}) > 0$ is

$$\{a \in \Phi \setminus pre(PF) \mid \delta_{PF}(q_{PF}, a) \text{ and } \delta_{SF}(q_{SF}, a) \text{ are defined } \} \subset \{a \in \Phi \setminus pre(PF) \mid \delta_{PF}(q_{PF}, a) \text{ and } \delta_{SF}(q'_{SF}, a) \text{ are defined } \} \quad (99b)$$

and if corresponding conditions w.r.t. SG and PG are fulfilled, then L_{IK} is self-similar.

In example 2 let $PF := \pi_\Phi(P)$ and $PG := \pi_\Gamma(P)$, then $\mathbb{PF}$ is given in Fig. 11(a) and $\mathbb{PG}$ is given in Fig. 12(a). In Sect. 5 we have shown that SF resp. SG is based deterministically on PF resp. PG w.r.t. $\mathbb{PF}$ resp. $\mathbb{PG}$. The product automaton of $\mathbb{SG}$ and $\mathbb{PG}^{\sqcup\!\sqcup}$ is given in Fig. 12(b).

To check the conditions of Theorem 12 w.r.t. SG and PG those pairs $[(q_{SG}, f), (q'_{SG}, f')]$ of states of the product automaton with $f \geq f'$ have to be considered. Let for example $(q_{SG}, f) = (4, \{(\text{III}, 1), (\text{II}, 1)\})$ and $(q'_{SG}, f') = (3, \{(\text{III}, 1)\})$. Then

$$\{a \in \Gamma \cap \text{pre}(PG) \mid \delta_{SG}(4, a) \text{ is defined}\} = \emptyset \subset \{\text{g}_\text{x}\} = \{a \in \Gamma \cap \text{pre}(PG) \mid \delta_{SG}(3, a) \text{ is defined}\}$$

and

$$\{q \in Q \mid f'(q) > 0\} = \{\text{III}\}.$$

Additionally

$$\{a \in \Gamma \setminus \text{pre}(PG) \mid \delta(\text{III}, a) \text{ and } \delta_{SG}(4, a) \text{ are defined}\} = \{\text{g}_\text{z}\} = \{a \in \Gamma \setminus \text{pre}(PG) \mid \delta(\text{III}, a) \text{ and } \delta_{SG}(3, a) \text{ are defined}\}.$$

Hence the conditions of Theorem 12 are fulfilled for the pair $[(4, \{(\text{III}, 1), (\text{II}, 1)\}), (3, \{(\text{III}, 1)\})]$. Analogously this can be shown for all other pairs with $f \geq f'$. It also can be proven that the conditions of Theorem 12 w.r.t. SF and PF are fulfilled.

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